Real-time game theoretic coordination of competitive mobility-on-demand systems

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Abstract—This paper considers competitive mobility-on-demand systems where a group of vehicle sharing companies provide pickup-delivery service in populated areas. The companies, on one hand, want to collectively regulate the traffic of the user queueing network, and on the other hand, aim to maximize their own net profit at each time instant. We formulate the strategic interconnection among the companies as a real-time game. We propose an algorithm to achieve vehicle balance and practical regulation of the user queueing network. We quantify the relation between the regulation error and the system parameters.

I. INTRODUCTION

Private automobiles are not a sustainable solution to personal mobility given their drawbacks of energy inefficiency, high greenhouse gas emissions and induced traffic congestion. Mobility-on-demand (MoD) systems represent a promising paradigm for future urban mobility. In particular, MoD systems are one-way vehicle-sharing systems where vehicle-sharing companies provide sharing vehicles at stations in a geographic region of interest, and users drive or are driven by the vehicles from a pickup location to a drop-off location. In MoD system, the arrivals and departures of users are uncorrelated, so it is important to real-time reallocate the vehicles to match the dynamic and spatial demands. In this paper, we focus on competitive MoD systems where multiple service suppliers compete with one another to maximize their own profits. The paper [14] instead studies the scenario where there is a single service supplier.

Literature review. Networked resource allocation among competing users has been extensively studied in the context of Game Theory. In [2], the authors exploit differential game theory to derive Nash equilibrium controllers for multiple self-interested users to manage the traffic of a single queue. However, the approach in [2] is not applicable to our problem due to: e.g, the additional dynamics of vehicle queues, the nonlinearity and non-smoothness of dynamic systems and the presence of state and input constraints. Static games have also been widely used to synthesize decentralized schemes [16] and best-response dynamics [11]. In [18], we consider distributed equilibrium computation over unreliable communication networks.

Contributions. In this paper, we present a model of competitive MoD systems and formulate the problem of real-time game theoretic coordination among multiple players (i.e., vehicle sharing companies). In particular, the players want to collectively regulate the traffic of the user queueing network through delivering the users to their destinations. On the other hand, each player aims to maximize his net profit at each time instant by maximizing the user delivery and minimizing the transition of empty vehicles. We propose an algorithm to achieve vehicle balance and practical regulation of the user queueing network. We quantify the relation between the regulation error and the system parameters (i.e., the maximum variation of the user arrival rates). For ease of presentation, the notations of Sections III and IV will be introduced and summarized in the Appendix. Due to the space limitation, we will omit the analysis and some discussion. The complete version is provided in [17].

II. PROBLEM FORMULATION

A. Model

A competitive MoD system consists of three interconnected networks: the user queueing network, the vehicle queueing network and the player network.

1) The user queueing network: There is a set of stations, say $\mathbb{S}$, in a spatial area of interest, and the interconnection of the stations is characterized by the graph $G_\mathbb{S} \equiv \{\mathbb{S}, \mathbb{E}_\mathbb{S}\}$ where $(\kappa, \kappa') \in \mathbb{E}_\mathbb{S} \setminus \text{diag}(\mathbb{S})$ if and only if the users at station $\kappa$ can be delivered to station $\kappa'$. The graph $G_\mathbb{S}$ is fixed, undirected and connected. Denote by $N_\kappa \equiv \{\kappa' \in \mathbb{S} \mid (\kappa, \kappa') \in \mathbb{E}_\mathbb{S}\}$ the set of neighboring stations of station $\kappa$.

Users arrive at station $\kappa \in \mathbb{S}$ in a dynamic fashion. Let $c_\kappa(t) \in \mathbb{R}_{>0}$ be the user arrival rate at station $\kappa$ at time $t$, and its temporal evolution is governed by the following ordinary differential equation:

$$\dot{c}_\kappa(t) = h_\kappa(c_\kappa(t), Q_\kappa(t), t).$$
In (1), $Q_κ(t)$ is the queue length of station $κ$ and will be defined later. The function $h_κ : R^3_{≥0} → R_{≥0}$ is the rate that the empty vehicles of player $i$ at station $κ$ and $κ’$ are transferred from station $κ$ to station $κ’$ at time $t$. Let $a_κ(κ’,t) ∈ [0,1]$ be the fraction of users who arrive at station $κ$ at time $t$ and want to reach station $κ’ ≠ κ$. Thus $\sum_{κ’∈S_κ} a_κ(κ’,t) = 1$. We assume that the fraction $a_κ(κ’,t)$ is fixed; i.e., $a_κ(κ’,t) = a_κ$ for $t ≥ 0$.

A queue is associated with each station $κ ∈ S$, and the arrived users wait for the delivery in the queue. Let $Q_κ(t) ∈ R_{≥0}$ be the queue length of station $κ$ at time $t$, and it dynamics is given by:

$$\dot{Q}_κ(t) = (c_κ(t) - u_κ(t))1_{[Q_κ(t) ≥ 0]},$$  \hspace{1cm} (2)

where $1$ is the indicator function and the initial state $Q_κ(0) > 0$, and the quantity $u_κ(t) = \sum_{i∈V} β_κ^i(t) ∈ R_{≥0}$ where $β_κ^i(t)$ is the delivery rate of player $i$ at station $κ$ and $V = \{1, ⋅, N\}$ is the set of players explained later.

Denote the vectors $Q ≜ (Q_κ(κ), κ∈S), c ≜ (c_κ(κ), κ∈S), ξ ≜ (ξ, κ) and u ≜ (u_κ), κ∈S$. We then rewrite (1) and (2) into the following compact form:

$$\dot{Q}(t) = h_Q(c(t), u(t)), \hspace{1cm} \dot{c}(t) = h_c(c(t), Q(t), t).$$ \hspace{1cm} (3)

and then,

$$\dot{ξ}(t) = h_ξ(ξ(t), u(t), t).$$ \hspace{1cm} (4)

2) The vehicle queueing network: There is a group of players $V ≜ \{1, ⋅, N\}$. Each player is a vehicle-sharing company, and he provides the service of delivering the users on the graph $G$. Let $v_κ^i(t) ∈ R_{≥0}$ be the number of vehicles of player $i$ stored at station $κ$ at time $t$. If $v_κ^i(t) > 0$, then player $i$ is able to deliver the users leaving station $κ$ at a rate $β_κ^i(t) ∈ [a, β_{κ,max}^i - a]$ with $0 < a < \frac{1}{2}β_{κ,max}^i$. Otherwise, player $i$ cannot deliver any user; i.e., $β_κ^i(t) = 0$. In order to avoid $v_κ^i(t)$ becoming zero, each player $i$ needs to reallocate his empty vehicles. Let $a_κ^i(κ) ∈ [a, α_{κ,max}^i - a]$ with $0 < a < \frac{1}{2}β_{κ,max}^i$ be the rate that the empty vehicles of player $i$ are transferred from station $κ$ to station $κ’$ at time $t$. The dynamics of $v_κ^i(t)$ is based on the mass-conservation law and given by:

$$\dot{v}_κ^i(t) = \{-β_κ^i(t) + \sum_{κ’∈N_κ} a_κ^i(κ’β_κ^i(κ')) t \}1_{[v_κ^i(t) > 0]}$$
$$\hspace{2cm} - \sum_{κ’∈N_κ} a_κ^i(κ’β_κ^i(κ')) t 1_{[v_κ^i(t) = 0]} \hspace{1cm} (5)$$

with the initial state $v_κ^i(0) > 0$. It is easy to verify that $v_κ^i(t) > 0$ for all $t ≥ 0$.

3) The player network: Each player in $V$ has three partially conflicting objectives: the first one is to collectively regulate the queue length $Q_κ(t)$ to the desired level $Q_κ ∈ R_{≥0}$, the second one is to maintain $v_κ^i(t)$ to be strictly positive, and the third one is to maximize his own net profits. In the sequel, we will explain each objective in more detail.

Firstly, players aim to collectively regulate the queue length $Q_κ(t)$ to the desired level $Q_κ$. For the time being, we assume that there exists a smooth controller $U(ξ(t)) ≜ (U_κ(ξ_κ(t)), κ∈S)$ which is able to achieve the queue regulation. Hence, players share a common goal of enforcing the controllers $u_κ(t) = U_κ(ξ_κ(t))$ for all $κ ∈ S$.

Secondly, each player $i$ wants to maintain $v_κ^i(t) > 0$ in order to sustain a non-trivial service rate $β_κ^i(t) > 0$. Since $v_κ^i(t) > 0$, player $i$ can achieve this goal through simply keeping $v_κ^i(t)$ as a constant; i.e., enforcing the hard constraint $v_κ^i(t) = 0$ all the time.

Thirdly, each player is self-interested and desires to maximize his net profit at each time instant. In particular, each player $i$ is able to make a profit from the delivery service, and the profit is modeled by $B_i(β_κ^i(t))$ where the function $B_i : R_{≥0} → R_{≥0}$ is smooth and strongly concave with constant $ρ_i > 0$. On the one hand, the transfer of empty vehicles is costly, and the expense is modeled by $C_i(α_κ^i(t))$ where $C_i : R_{≥0} → R_{≥0}$ is smooth and strongly convex with constant $ρ_i > 0$. The net cost of player $i$ at time $t$ is abstracted by $\sum_{κ∈S} C_i(α_κ^i(t)) - \sum_{κ∈S} B_i(β_κ^i(t))$.

The decision vector of player $i$ at time $t$ is given by $z_κ^i(t)$ which is the collection of $α_κ^i(t) ≜ (α_κ^i(t), (κ’), κ∈S)$ and $β_κ^i(t) ≜ (β_κ^i(t), κ∈S)$. The above three interests of player $i$ at time $t$ are compactly expressed by the following convex program parameterized by the vector $U(ξ(t))$:

$$\min_{z_κ^i(t) ∈ R_{≥0}} \sum_{κ, κ’∈S} C_i(α_κ^i(t)) - \sum_{κ∈S} B_i(β_κ^i(t)),$$

s.t. $\sum_{κ∈S} β_κ^i(t) = U_κ(ξ_κ(t)), \hspace{1cm} κ ∈ S,$

$$- β_κ^i(t) + \sum_{κ’∈N_κ} a_κ^i(κ’β_κ^i(κ')) t - \sum_{κ’∈N_κ} a_κ^i(κ’β_κ^i(κ')) t 1_{[v_κ^i(t) = 0]} \hspace{1cm} (6)$$

where the dimension $n_i = |S| + 2|E|$ and the set $Z_i$ is defined as: $Z_i ≜ \{z_i | β_κ^i ∈ [a, β_{κ,max}^i - a], \hspace{1cm} κ ∈ S, \hspace{1cm} α_κ^i ∈ [a, α_{κ,max}^i - a], \hspace{1cm} (κ, κ’) ∈ E_i\}$. In (6), the decisions of the players are coupled via the constraint $\sum_{κ∈S} β_κ^i(t) = U_κ(ξ_κ(t))$ which represents the common goal of the queue regulation. Other components in (6) are insteadarable.

By using $v(x) = 0$ if and only if $v(x) ≤ 0$ and $v(x) ≥ 0$, we rewrite the parametric convex program (6) into the following compact form:

$$\min_{z_κ^i(t) ∈ R_{≥0}} f_i(z_κ^i(t)),$$

s.t. $G_i(β_κ^i(t), β_κ^i(t), U(ξ(t))) ≤ 0,$

$$h_i(z_κ^i(t)) ≤ 0, \hspace{1cm} z_κ^i(t) ∈ Z_i,$$ \hspace{1cm} (7)

where $G : R^n×|S| → R^m$ ($m = 2|S|$) and $h_i : R^n → R^p$ ($p = 2|S|$) are affine functions. The components of $G$ and $h_i$ are asymmetric; i.e., $G_κ = -G_κ+|S|$ and $h_κ^i = -h_κ^i+|S|$.
for $1 \leq \ell \leq |S|$. The collection of (7) will be referred to as the CVX game parameterized by $U(\xi(t))$, and its solution, Nash equilibrium, is defined as follows:

**Definition 2.1:** For the CVX game parameterized by $U(\xi(t))$, the state $\tilde{z}(t) \in Z \triangleq \prod_{i \in V} Z_i$ is a Nash equilibrium if and only if:

1. $G(\tilde{\beta}(t), U(\xi(t))) \leq 0$ and $h^{[i]}(\tilde{z}^{[i]}(t)) \leq 0$;
2. for any $z^{[i]} \in Z_i$ with $G(\beta^{[i]}, \tilde{\beta}^{[i]}(t), U(\xi(t))) \leq 0$ and $h^{[i]}(z^{[i]}) \leq 0$, it holds that $f_i(\tilde{z}^{[i]}(t)) \leq f_i(z^{[i]})$.

Since $f_i$ is strongly convex and separable, the map of partial gradients is strongly monotone, and thus the set of Nash equilibria $X_C(\xi(t))$ is non-empty; e.g., in [8].

**B. Our objective**

At each time instant $t$, the players aim to solve the CVX game parameterized by the control command $U(\xi(t))$, and implement a Nash equilibrium in $X_C(\xi(t))$. This procedure is repeated at the next time instant by shifting the time horizon forward. We term the collection of these instantaneous games over the infinite horizon as a real-time game. In this paper, we will design an algorithm to update $z(t)$ such that real-time game theoretic coordination is achieved; that is,

$$\lim_{t \to +\infty} \text{dist}(z(t), X_C(\xi(t))) = 0,$$

$$\lim_{t \to +\infty} Q_k(t) = \bar{Q}_k, \quad k \in S,$$

$$v^{[i]}_k(t) = v^{[i]}_k(0), \quad k \in S, \quad i \in V, \quad t \geq 0. \quad (8)$$

**Remark 2.1:** In contrast to [2], [4], our real-time game theoretic coordination formulation relaxes the computation of infinite-horizon Nash equilibrium. Instead, our formulation aims to real-time seek the collection of instantaneous Nash equilibrium. The formulation allows us to handle constrained discontinuous dynamic systems and relax the *a priori* information of the user arrival rates over the infinite horizon.

Our game formulation is partially motivated by receding-horizon control or model predictive control; e.g., in [10], whose control laws are based on solving a sequence of finite horizon optimal control problems. Our game formulation is also relevant to optimization and games in dynamic environments; e.g., in [5], [6]. However, this set of papers only consider open-loop decision making.

**C. Assumptions**

Let $\beta_{\max} \triangleq \sum_{i \in V} \beta_{\max}^{[i]}, \alpha_{\max} \triangleq \sum_{i \in V} \alpha_{\max}^{[i]}$. In the remainder of this paper, we suppose that the following set of assumptions hold.

**Assumption 2.1:** It holds that $\beta_{\max}^{[i]}$ are identical for all $i \in V$ and $c_k(t) \in [c_{\min}, c_{\max}]$ for all $t \geq 0$ and $k \in S$. In addition, $2Na \leq c_{\min} < c_{\max} < \beta_{\max} - Na$.

**Assumption 2.2:** There is $\delta > 0$ such that $\|\dot{c}_k(t)\| \leq \delta_c$ for all $k$ and $t \geq 0$.

**Assumption 2.3:** For any $\beta^{[i]}$ with $\beta_{\max}^{[i]} \in [a, \beta_{\max}^{[i]} - a]$, there is $\alpha_{\max}^{[i]}$ such that $\alpha_{\max}^{[i]} \in [a, \alpha_{\max}^{[i]} - a]$ and the following holds for $k \in S$:

$$-\beta^{[i]}_k + \sum_{k' \in N_k} a_{k', k} \beta^{[i]}_{k'} - \sum_{k' \in N_k} a_{k', k} \alpha_{k'}^{[i]} + \sum_{k' \in N_k} a_{k', k} \alpha_k^{[i]} = 0.$$

The combination of Assumptions 2.1 and 2.2 implies that there is $\delta_c > 0$ such that $\|\xi(t)\| \leq \delta_c$ for all $t \geq 0$.

**III. Preliminaries**

In the sequel, we will first present a smooth control law to regulate the user queues. Then we will introduce an approximation of the CVX game parameterized by $\zeta(t) \triangleq U(\xi(t))$, namely, the regularized game. In order to simplify the notations, we will drop the dependency of $\zeta(t)$ on time $t$. After this, we will perform sensitivity analysis on the regularized game.

**A. The existence of smooth controllers**

With Assumption 2.1, we will show that the regulation of user queues can be achieved via the following $u_k$:

$$u_k(t) = U_k(\xi(t)) = c_k(t) - \hat{U}_k(Q_k(t)) \triangleq c_k(t) - c_{\min} + \frac{\beta_{\max} - c_{\max} + c_{\min} - Na}{1 + \frac{2}{\beta_{\max} - c_{\max} - Na}} (Q_k(t) - Q_k(t)).$$

Towards this end, it is easy to verify that $\hat{U}_k(Q_k) = c_k(t)$ and $U_k(Q_k(t)) \in [c_{\min}, \beta_{\max} - Na] \subseteq [Na, \beta_{\max} - Na]$ by utilizing the monotonicity of the functions in $U$. Hence, the controller $U_k(\xi(t))$ is realizable for the players. Furthermore, $\frac{1}{2} \partial^2 U_k(Q_k(t) - Q_k(t)) = (Q_k(t) - Q_k(t)) \partial U_k(Q_k(t)) < 0$ for $Q_k(t) < Q_k(t)$ and $Q_k(t) > 0$. Hence, $U_k(\xi(t))$ is able to regulate $Q_k(t)$ to $Q_k$ from any initial state $Q_k(0) > 0$. This controller will be used in the remainder of the paper. The following lemma finds uniform upper bounds on $\|dU(\xi(t))\|$ and $\|d^2U(\xi(t))\|$.

**Lemma 3.1:** The following holds for all $\xi \geq 0$:

$$\|\frac{dU(\xi)}{d\xi}\| \leq D^{(1)}, \quad \|\frac{d^2U(\xi)}{d\xi^2}\| \leq D^{(2)}. \quad (9)$$

**B. The regularized game**

1) Regularized Lagrangian functions: To relax the constraints of $G(\beta, \zeta) \leq 0$ and $h^{[i]}(z^{[i]}) \leq 0$, we define the following regularized Lagrangian function for player $i$:

$$L_i(z, \mu, \lambda^{[i]}, \zeta) = f_i(z^{[i]}) + \langle \mu, G(\beta, \zeta) \rangle + \langle \lambda^{[i]}, h^{[i]}(z^{[i]}) \rangle - \tau \sum_{k \in S} \psi(\beta_{\max}^{[i]} - \beta_k^{[i]})$$

$$- \tau \sum_{(k, k') \in E_2} \left( \psi(\alpha_k^{[i]} - \alpha_{k'}^{[i]}) + \psi\left(\alpha^{[i]}_{\max} - \alpha_k^{[i]}\right) \right) - \frac{\lambda^{[i]}}{2} \|\mu\|^2 - \frac{\lambda^{[i]}}{2} \|\lambda^{[i]}\|^2 + \tau \sum_{t=1}^{m} \psi(\mu^t) + \tau \sum_{\ell=1}^{p} \psi(\lambda^{[i]}_{\ell}), \quad (10)$$

with $\epsilon > 0$, $\tau > 0$ and $\mu \in R^m$ and $\lambda^{[i]} \in R^p$ are dual multipliers. The function $\psi$ is the logarithmic barrier function and defined as follows:

$$\psi(s) = \log\left(\frac{s}{s}\right), \quad s > 0,$$

$$\psi(s) = -\infty, \quad s \leq 0.$$

Note that $\psi$ is concave and monotonically increasing over $R_{>0}$. In $L_i$, the hard constraints $\mu_t \geq 0, \lambda^{[i]}_t \geq 0$ and
\[ z^i \in Z_i \] are relaxed by those defined via the logarithmic function. In addition, the terms associated with \( \epsilon \) play a role of regularization as shown in Lemma 3.2.

We then introduce a set of dual players \( \{0\} \cup V_m \triangleq \{1, \ldots , N\} \), and \( \mu \) is the decision vector of dual player 0, and \( \lambda^i \) is the decision vector of dual player \( i \). Each primal player \( i \in V \) aims to minimize \( L_i \) over \( z^i \in R^{m_i} \). Each dual player \( i \in V_m \) desires to maximize \( L_i \) over \( \lambda_i \in R^p \) and dual player 0 wants to maximize \( H(\mu, \lambda, \zeta) \triangleq (\mu, G(\beta, \zeta)) - \frac{\tau}{2} \|\mu\|^2 + \tau \sum_{e=1}^{N} \psi(\mu_e) \). This game is referred to as the regularized game (RG game, for short) parameterized by \( \zeta \) and the definition of its NEs is given as follows:

**Definition 3.1:** The state \( (\tilde{z}, \tilde{\mu}, \tilde{\lambda}) \in R^{n+m+p} \) is a Nash equilibrium of the RG game parameterized by \( \zeta \) if and only if the following hold for each primal player \( i \):

\[ L_i(\tilde{z}, \tilde{\mu}, \tilde{\lambda}, \zeta) \leq L_i(z^i, \tilde{z}^{i-}, \tilde{\mu}, \tilde{\lambda}^i, \zeta), \quad \forall z^i \in R^{m_i}, \]

the following hold for each dual player \( i \in V_m \):

\[ L_i(\tilde{z}, \tilde{\mu}, \lambda^i, \zeta) \leq L_i(\tilde{z}, \tilde{\mu}, \tilde{\lambda}^i, \zeta), \quad \forall \lambda^i \in R^p, \]

and the following hold for dual player 0:

\[ H(\tilde{z}, \tilde{\mu}, \zeta) \leq H(\tilde{z}, \tilde{\mu}, \zeta), \quad \forall \mu \in R^m. \]

The set of NEs of the RG game parameterized by \( \zeta \) is denoted as \( X_{\text{RG}}(\zeta) \). Since the logarithmic barrier function \( \psi \) penalizes \( \mu \not\in R^{m_0} \) an infinite cost, thus it must be \( \tilde{\mu}(\zeta) > 0 \) for any NE \( \tilde{\eta}(\zeta) \in X_{\text{RG}}(\zeta) \). Analogously, \( \lambda^i(\zeta) > 0, \beta^i(\zeta) \in (0, \beta^i_{\text{max}}] \), and \( \delta^i(\zeta) \in (0, \delta^i_{\text{max}}] \) for any NE \( \tilde{\eta}(\zeta) \in X_{\text{RG}}(\zeta) \).

2) **Convexity of the RG game:** Since \( f_i \) is strongly convex in \( z^i \) and \( \psi \) is concave over \( R \), then \( L_i \) is strongly convex in \( z^i \in Z_i \triangleq \{z^i \in R^{m_i} \mid \|z^i\| \leq \beta^i_{\text{max}} \}, \kappa \in S_i, \alpha^i_{\text{max}} \in [0, \alpha^i_{\text{max}}], (\kappa, \kappa') \in E_i \} \) with constant \( \min(\alpha_0, \alpha') \). By introducing the quadratic perturbation of \( \frac{\tau}{2} \|\lambda^i\|^2 \), the function of \( L_i(\tilde{z}, \tilde{\mu}, \cdot) \) is strongly concave in \( \lambda^i \) with constant \( \epsilon \) over \( R^{p_0} \). This can be verified by the following computation:

\[ \frac{d^2 L_i}{d(\lambda^i)^2} = -\epsilon - \frac{\tau}{(\lambda^i)^2} < -\epsilon. \]

Analogously, the function of \( H(\tilde{z}, \mu) \) is strongly concave in \( \mu \) with constant \( \epsilon \) over \( R^{m_0} \).

3) **Monotonicity of the RG game:** It is noted that all the functions involved in \( L_i \) are smooth in \( Z \times R^{m_0} \times R^{p_0} \). We then define \( \nabla_{z^i} L_i(z, \mu, \lambda^i, \zeta) : Z \times R^{m_0} \times R^{p_0} \rightarrow R^{m_i} \) as the partial gradient of the function \( L_i(z, \mu, \lambda^i, \zeta) \) at \( z^i \). Other partial gradients can be defined in an analogous way. Let \( \eta \triangleq (\mu, \lambda, \zeta) \), and define the map \( \nabla \Omega : Z \times R^{m_0} \times R^{p_0} \rightarrow R^{n+m+p} \) as partial gradients of the player's objective functions:

\[ \nabla \Omega(\eta, \zeta) \triangleq \left[ \nabla_{z^i} L_1(z, \mu, \lambda^1, \zeta)^T \cdots \nabla_{z^i} L_N(z, \mu, \lambda^N, \zeta)^T \right] = \left[ \begin{array}{c} \nabla H(z, \mu, \zeta)^T \\nabla_{\lambda^i} L_1(z, \mu, \lambda^i, \zeta)^T \cdots \nabla_{\lambda^i} L_N(z, \mu, \lambda^N, \zeta)^T \end{array} \right]^T. \]

The following lemma shows that the quadratic perturbations of \( \frac{\tau}{2} \|\lambda\|^2 \) and \( \frac{\tau}{2} \|\lambda^i\|^2 \) regularize the game map \( \nabla \Omega \) to be strongly monotone over \( \tilde{Z} \times R^{m_0+p} \).

**Lemma 3.2:** The regularized game map \( \nabla \Omega(\eta, \zeta) \) is strongly monotone over \( \tilde{Z} \times R^{m_0+p} \) with constant \( \rho_0 = \min_{\eta \in V} \{\rho_0, \rho'_1, \epsilon\} \). In addition, there is a unique NE \( \tilde{\eta}(\zeta) \in X_{\text{RG}}(\zeta) \).

**C. Sensitivity analysis**

In this part, we will drop the dependency of the NE \( \tilde{\eta}(\zeta) \) on \( \zeta \) unless necessary. As mentioned before, in the RG game, the hard constraints \( \mu_0 \geq 0, \lambda^i_{\text{min}} \geq 0 \), and \( z^i \in Z_i \) are relaxed by those defined via the logarithmic function. Hence, the RG game is completely unconstrained. This allows us to perform sensitivity analysis on the RG game, and characterize the variation of the NE \( \tilde{\eta}(U(\xi)) \) induced by the variation of \( \xi \). On the other hand, in the enlarged version [17], we show the RG game can be rendered arbitrarily close to the CVX game by choosing a pair of sufficiently small \( \epsilon \) and \( \tau \).

By the convexity or concavity of \( L_i \) on its components, the following first-order conditions hold for the unique NE \( \tilde{\eta}(\zeta) \in X_{\text{RG}}(\zeta) \):

\[ \nabla_{z^i} L_i(z, \tilde{\mu}, \tilde{\lambda}^i, \zeta) = 0, \quad \nabla_{\lambda^i} L_i(z, \tilde{\mu}, \tilde{\lambda}^i, \zeta) = 0, \quad \nabla_{\mu} H(z, \tilde{\mu}, \zeta) = 0. \]

(12)

Toward this end, we denote a set of matrices as follows:

\[ R_1(\tilde{\eta}) \triangleq \begin{bmatrix} \nabla_{z^i} L_1(z, \tilde{\mu}, \tilde{\lambda}^i, \zeta) & \cdots & \nabla_{z^i} L_N(z, \tilde{\mu}, \tilde{\lambda}^i, \zeta) \end{bmatrix}, \]

\[ R_2(\tilde{\eta}) \triangleq \begin{bmatrix} \nabla_{z^i} G_1(\beta, \zeta)^T & \cdots & \nabla_{z^i} G_m(\beta, \zeta)^T \end{bmatrix}, \]

\[ R_3(\tilde{\eta}) \triangleq \text{diag}(\nabla_{z^i} h^i_{\text{min}}(z^i)^T), \cdots, \nabla_{z^i} h^i_{\text{max}}(z^i)^T) \] \text{in } V.

Recall that \( G \) and \( h^i \) are affine. Then \( \nabla_{z^i} L_i = 0 \) if \( i \neq j \). Since \( L_i \) is separable in its components, thus \( R_3(\tilde{\eta}, \zeta) \) is diagonal, symmetric and positive definite. In addition, \( R_2 \) and \( R_3 \) are constant due to \( G \) and \( h^i \) being affine.

With the above notations at hand, we can derive the partial derivative of the left-hand side of (12) with respect to \( \tilde{\eta} \) evaluated at \( (\tilde{\eta}, \zeta) \), and this derivative is given by:

\[ J_M(\tilde{\eta}) \triangleq \begin{bmatrix} R_1(\tilde{\eta}) & R_2(\tilde{\eta}) & R_3(\tilde{\eta}) \\ -R_2 & R_4(\tilde{\eta}) & 0 \\ -R_3 & 0 & R_5(\tilde{\eta}) \end{bmatrix}. \]

Let \( J_N \) be the partial derivative of the left-hand side of (12) with respect to \( \zeta \). Since \( G \) is affine in \( \zeta \), then \( J_N \) is state-independent. We then denote

\[ J(\tilde{\eta}) \triangleq J_M(\tilde{\eta})^{-1} J_N. \]

(13)
where \( J_M(\tilde{\eta})^{-1} \) will be shown to be non-singular in the following lemma.

**Lemma 3.3:** The matrix \( J_M(\tilde{\eta}(\zeta)) \) is non-singular, positive definite and its spectrum is uniformly lower bounded by \( \epsilon \min_{i \in V} \{p_i, p'_i\} > 0 \). In addition, \( J(\tilde{\eta}(\zeta)) \) is continuously differential in \( \zeta \), and the following relation holds:

\[
\frac{d\tilde{\eta}(U(\xi(t)))}{dt} = J(\tilde{\eta}(U(\xi(t)))) \frac{dU(\xi(t))}{d\xi(t)} \tilde{\xi}(t).
\]

**Remark 3.1:** In the paper [6], a relation like (14) between saddle-points and the parameter is derived from the Karush-Kuhn-Tucker condition. However, the results in [6] are not applicable to our problem. Firstly, the Lagrangian functions \( \mathcal{L} \) and \( \mathcal{H} \) are merely concave in \( \lambda^{[i]} \) and \( \mu \) if \( \epsilon, \tau = 0 \). Secondly, the paper [6] assumes that the state-dependent matrix derived from the Karush-Kuhn-Tucker condition is uniformly non-singular. This is not easy to check a priori and may lead to instability in our feedback setup.

**Lemma 3.4:** The functions \( J(\eta) \frac{dt}{d\eta} \) and \( \nabla \Omega \) are Lipschitz continuous with constant \( L_J > 0 \) and \( L_\Omega > 0 \), respectively, over \( Y \).

IV. REAL-TIME GAME THEORETIC COORDINATION

In this section, we will present an algorithm for the real-time game theoretic coordination. It will be followed by the convergence properties of the closed-loop system.

**A. Algorithm statement**

The real-time game theoretic regulator is stated in Algorithm 1. In the algorithm, \( (\alpha^{[i]}(t), \beta^{[i]}(t)) = Q_i(\tilde{\eta}^{[i]}(t)) \) where \( \beta^{[i]}(t) = \beta_0^{[i]}(t) \) and \( \alpha^{[i]}(t) \) is the orthogonal projection of \( \alpha^{[i]}(t) \) onto the set \( \Xi^{[i]}(t) \) defined by:

\[
\Xi^{[i]}(t) \triangleq \{ \alpha^{[i]} \in \mathbb{R}^{\mathcal{E}_i} \mid A\alpha^{[i]} = b^{[i]}(t), \alpha^{[\kappa \kappa']}_i \in [a, \alpha^{[i]}_{\max} - a], (\kappa, \kappa') \in \mathcal{E}_i \}
\]

where \( b^{[i]}_\kappa(t) = \beta_0^{[i]}(t) - \sum_{\kappa' \in \mathcal{N}_e} a^{[\kappa \kappa']}_i \beta^{[i]}_\kappa'(t) \) and \( b^{[i]}(t) = (b^{[i]}_\kappa(t))_{\kappa \in \mathcal{E}_i} \).

**B. Performance analysis**

The closed-loop system consists of the user queueing network (3), the vehicle queueing network (5) and the real-time game theoretic coordinator (Algorithm 1).

The following theorem summarizes the performance of the closed-loop system.

**Theorem 4.1:** Suppose Assumptions 2.1, 2.2 and 2.3 hold. Suppose the following holds:

\[
\vartheta \triangleq (1 - \alpha \rho_1 + \alpha^2 L_1^2 + (L_J D_u^{(1)}(\delta)) \frac{1}{\delta} < 1.
\]

The estimates \( \beta^{[i]}(t) \) and \( \alpha^{[i]}(t) \) generated by Algorithm 1 approximates \( \tilde{\eta}(t) = X_{RG}(\zeta(t)) \) in the way that

\[
\lim_{t \to +\infty} ||\beta^{[i]}(t) - \beta_0^{[i]}(t)|| = 0, \\
\lim_{t \to +\infty} ||\alpha^{[i]}(t) - \alpha_0^{[i]}(t)|| \leq \|A\|s_G(\epsilon, \tau),
\]

Furthermore, the queue dynamics achieve the following:

\[
\lim_{t \to +\infty} \|Q_i(t) - \hat{Q}_i\| \leq \max\{\Delta_{\min}, \Delta_{\max}\},
\]

where \( \Delta_{\min} \triangleq \ln(1 + \frac{2\vartheta(\epsilon, \tau)}{\beta^{[\max^{\max}} - \alpha_{\min}(\delta)Q_i(\epsilon)}) \) and \( \Delta_{\max} \triangleq -\ln(1 - \frac{\vartheta(\epsilon, \tau)}{\beta^{[\max^{\max}} - \alpha_{\min}(\delta)Q_i(\epsilon)}) \).

**Remark 4.1:** Recall that \( \rho_1 = \min_{i \in \mathcal{E}} \{\rho_i, \rho_i', \epsilon\} \) and \( \delta \) represents an upper bound on the maximum variation of the user arrival rate. If \( \delta \) is smaller, we can choose a set of smaller \( \alpha, \epsilon \) and \( \tau \) to satisfy \( \delta < 1 \) and reduce the right-hand sides of (15) and (16). That is, if the user arrival rates change slower, the steady-state system performance can be improved.

V. CONCLUSIONS

In this paper, we have introduced a model of competitive MoD systems and proposed a real-time game theoretic coordination problem for the system. We have came up with an algorithm to achieve vehicle balance and practical regulation of the user queueing network.

VI. APPENDIX

A. Notations for Section III

Denote \( Z_i \triangleq \{z^{[i]} \in \mathbb{R}^{n_i} \mid |\beta^{[i]}_\kappa| \in [a, \beta^{[i]}_{\max} - a], \kappa \in \mathcal{S}_i, \alpha^{[\kappa \kappa']}_i \in [a, \alpha^{[i]}_{\max} - a], (\kappa, \kappa') \in \mathcal{E}_i\} \), \( Z_i \triangleq \bigcup_{i \in \mathcal{E}^V} Z_i, \)

\[
Z_i \triangleq \{z^{[i]} \in \mathbb{R}^{n_i} \mid |\beta^{[i]}_\kappa| \in [0, \beta^{[i]}_{\max}], \kappa \in \mathcal{S}_i, \alpha^{[\kappa \kappa']}_i \in [0, \alpha^{[i]}_{\max}], (\kappa, \kappa') \in \mathcal{E}_i\} \text{ and } Z \triangleq \bigcup_{i \in \mathcal{E}} Z_i.
\]
\[
D_U^{(1)} \triangleq |S|(1 + \frac{\beta_{\text{max}} - c_{\text{max}} + c_{\text{min}} - Na}{2 - Na}(1 + 2\frac{\beta_{\text{max}} - c_{\text{max}} - Na}{c_{\text{min}}})^2)\varepsilon_{\text{max},e_S}^2 \phi_S^2,
\]
\[
D_U^{(2)} \triangleq \frac{(\beta_{\text{max}} - c_{\text{max}} + c_{\text{min}} - Na)^2}{2 - Na}(1 + 2\frac{\beta_{\text{max}} - c_{\text{max}} - Na}{c_{\text{min}}})^2 \times (1 + (2\frac{\beta_{\text{max}} - c_{\text{max}} - Na}{c_{\text{min}}} + \frac{c_{\text{min}}}{\varepsilon_{\text{max},e_S}^2} Q_S^2))\varepsilon_{\text{max},e_S}^2 \phi_S^2.
\]
\[
g(s) \triangleq \frac{1}{2\varepsilon}(s + \sqrt{s^2 + 4\varepsilon}),
\]
\[
\delta_i \triangleq \sup_{z[i] \in Z_i} f_i(z[i]) - \inf_{z[i] \in Z_i} f_i(z[i]) + 2\varepsilon(|\varepsilon|\psi(\beta_{\text{max}}) + |\varepsilon_S|\psi(\alpha_{\text{max}})),
\]
\[
\varsigma(h, \tau) \triangleq \max_{\tau \in \varepsilon} \sqrt{N\varepsilon(\delta_i + (p - 1)\tau)},
\]
\[
\varsigma_G(h, \tau) \triangleq \max_{\tau \in \varepsilon} \sqrt{N\varepsilon(\delta_i + p\tau)},
\]
\[
\delta_i' \triangleq \sup_{z[i] \in Z_i} \|\nabla_{\alpha_{\text{max}}} f_i(z[i])\| + 2|\varepsilon|\max_{\tau \in \varepsilon} \{g(-\varsigma_G(h, \tau), g(\varsigma_G(h, \tau))\}
\]
\[
\delta_i'' \triangleq \sup_{z[i] \in Z_i} \|\nabla_{\alpha_{\text{max}}} f_i(z[i])\| + 2|\varepsilon|\max_{\tau \in \varepsilon} \{g(-\varsigma_G(h, \tau), g(\varsigma_G(h, \tau))\},
\]
\[
\Delta_\mu \triangleq \varepsilon \varsigma_G(h, \tau) + \min_{\tau \in \varepsilon} \min_{\mu \in \varepsilon} \|J_N\|D_u^{(1)}(\delta_i),
\]
\[
\Delta_\lambda \triangleq g(\varsigma_G(h, \tau)) + \min_{\tau \in \varepsilon} \min_{\lambda \in \varepsilon} \|J_N\|D_u^{(1)}(\delta_i),
\]
\[
d_\beta \triangleq \max_{\tau \in \varepsilon} \sup_{z[i] \in Z_i} \|\nabla_{\beta_{\text{max}}} f_i(z[i])\| + 2|\varepsilon|(|\Delta_\mu| + \Delta_\lambda),
\]
\[
d_\alpha \triangleq \max_{\tau \in \varepsilon} \sup_{z[i] \in Z_i} \|\nabla_{\alpha_{\text{max}}} f_i(z[i])\| + 2|\varepsilon|(|\Delta_\mu| + \Delta_\lambda),
\]
\[
d_\beta \triangleq \frac{\tau_{\beta_{\text{max}}}}{2\tau + d_\beta_{\beta_{\text{max}}}}, \quad d_\alpha \triangleq \frac{\tau_{\alpha_{\text{max}}}}{2\tau + d_\alpha_{\alpha_{\text{max}}}},
\]
\[
L_J \triangleq \sqrt{\left(\sup_{\eta \in \varepsilon} \left\|\frac{\partial R_1(\eta)}{\partial \eta}\right\|^2 + 4\varepsilon^2 \left(\frac{\varepsilon_{\beta_{\text{max}}}^2}{\varepsilon_{\alpha_{\text{max}}}^2}(\varepsilon, \tau) + \frac{\varepsilon_{\beta_{\text{max}}}^2}{\varepsilon_{\alpha_{\text{max}}}^2}(\varepsilon, \tau)\right)\right)}
\times \min_{\tau \in \varepsilon} \min_{\mu \in \varepsilon} \min_{\lambda \in \varepsilon} \|J_N\|D_u^{(1)} + \min_{\tau \in \varepsilon} \min_{\mu \in \varepsilon} \|J_N\|D_u^{(2)}.
\]

B. Notations for Section IV

We associate the incidence matrix A ∈ R^[|S| \times |2|]|S| for the graph G_S. In particular, the k-th row is assigned to state k, and is in the form of [a_{k1}, \ldots, a_{kn}], a_{ki} \in \{0, 1\}. If k' \not\in N_k, then \alpha_{k,k'} = -1; if k' \in N_k, then \alpha_{k,k'} = 1; \alpha_{k,k'} = 0, otherwise. Let \Lambda \triangleq \{\lambda[i] \in R^n | \lambda[i] \in \{0, \Delta_k\}, \forall k \in S\}, and M \triangleq \{\mu \in R^m | \mu \in \{0, \Delta_k\}, \forall k \in S\}.

REFERENCES


