Brief paper

Discrete-time dynamic average consensus

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A R T I C L E   I N F O

Article history:
Received 23 July 2008
Received in revised form 1 July 2009
Accepted 8 October 2009
Available online 24 November 2009

Keywords:
Consensus algorithms
Multi-agent systems
Cooperative control

A B S T R A C T

We propose a class of discrete-time dynamic average consensus algorithms that allow a group of agents to track the average of their reference inputs. The convergence results rely on the input-to-output stability properties of static average consensus algorithms and require that the union of communication graphs over a bounded period of time be strongly connected. The only requirement on the set of reference inputs is that the maximum relative deviation between the n-th-order differences of any two reference inputs be bounded for some integer n ≥ 1.

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1. Introduction

We consider the problem in which a set of autonomous agents aims to track the average of individually measured time-varying signals by local communication with neighbors. This problem is referred to as dynamic average consensus in opposition to the more studied static consensus. The dynamic average consensus problem arises in different contexts, such as formation control (Yang, Freeman, & Lynch, 2008), sensor fusion (Olfati-Saber, submitted for publication; Olfati-Saber & Shamma, 2005; Spanos, Olfati-Saber, & Murray, 2005), distributed estimation (Martínez, 2007) and distributed tracking (Yang, Freeman, & Lynch, 2007). These tasks require that all agents agree on the average of time-varying signals and thus the consensus on a static average value, e.g., the initial states of the agents, is insufficient.


The convergence rate of consensus algorithms is e.g., discussed in Olshevsky and Tsitsiklis (2009), Xiao and Boyd (2004), consensus propagation is considered in Moallemi and Roy (2006), and conditions on consensus algorithms to achieve different consensus values is discussed in Cortés (2006). Consensus algorithms find application in a variety of areas such as load balancing (Cybenko, 1989; Xu & Lau, 2007), formation control (Fax & Murray, 2004; Yang et al., 2008), and, as we have mentioned, sensor fusion (Martínez, 2007; Olfati-Saber, submitted for publication; Olfati-Saber & Shamma, 2005; Spanos et al., 2005), distributed tracking (Olfati-Saber, 2007; Yang et al., 2007) and consensus-based belief propagation in Bayesian networks (Olfati-Saber, Franco, Frazzoli, & Shamma, 2006).

The dynamic average consensus problem in continuous-time is studied in Freeman, Yang, and Lynch (2006), Olfati-Saber and Shamma (2005), Ren (2007) and Spanos et al. (2005). By using standard frequency-domain techniques, the authors in Spanos et al. (2005) showed that their algorithm was able to track the average of ramp reference inputs with zero steady-state error. In the context of input-to-state stability, Freeman et al. (2006) showed that proportional dynamic average consensus algorithm could track with bounded steady-state error the average of bounded reference inputs with bounded derivatives. On the other hand, Freeman et al. (2006) showed that proportional-integral dynamic average consensus algorithm could track the average of constant reference inputs with sufficiently small steady-state error.
error. The authors in Olfati-Saber and Shamma (2005) proposed a dynamic consensus algorithm and applied it to the design of consensus filters. The algorithm in Olfati-Saber and Shamma (2005) can track with some bounded steady-state error the average of a common reference input with a bounded derivative. The problem studied in Ren (2007) is similar to that in Olfati-Saber and Shamma (2005), and consensus of agents is over a common time-varying reference signal. However, the algorithm in Ren (2007) assumes that agents know the nonlinear model which generates the time-varying reference function. The problem studied in the present paper is close to those in Freeman et al. (2006) and Spanos et al. (2005) and includes those in Olfati-Saber and Shamma (2005) and Ren (2007) as special cases.

Statement of contributions. In this paper, we propose a class of discrete-time dynamic average consensus algorithms and analyze their convergence properties. This paper contributes to the problem of dynamic average consensus in the following aspects: The continuous-time communication assumption for dynamic average consensus in Freeman et al. (2006) and Spanos et al. (2005) is relaxed, and we consider more realistic discrete-time synchronous communication models. This allows us to obtain a direct relation between the frequency of inter-agent communication and the differences of reference signals. Our algorithms are able to track the average of a larger class of time-varying reference inputs than Freeman et al. (2006) and Spanos et al. (2005) with zero or sufficiently small steady-state error. This includes polynomials, logarithmic-type functions, periodic functions and other functions whose nth-order differences are bounded, for \( n \geq 1 \). We can also handle the case where the reference is the common part, that appears in all the individual reference inputs, explodes. Furthermore, the algorithms proposed are robust to the dynamic change of communication topologies as well as the joining and leaving (or failure) of nodes. The convergence analysis for our dynamic average consensus algorithms relies upon the input-to-output stability property of discrete-time static average consensus algorithms in the presence of external disturbances.

Organization of the paper. We now outline the remainder of the paper. In Section 2, we introduce general notation and the statement of the problem we study. In Section 3, we focus on a first-order algorithm for dynamic average consensus. Section 4 generalizes this to a class of nth-order algorithms for dynamic average consensus and analyze their convergence properties. In Section 5, we present some remarks on the extension of the results in Sections 3 and 4. In Section 6, we provide some examples to illustrate the effectiveness of our algorithms. Section 7 includes some concluding remarks. The proofs of our main results are given in the Appendix. The enlarged version of this paper can be found on Zhu and Martinez (2008).

2. Preliminaries and problem statement

In this section, we introduce the notation to be employed along the paper and state the problem of dynamic average consensus.

The positive real number \( h \) is the time discretization unit and the update time instants \( t \in \mathbb{R} (or \mathbb{R}^+) \) will be of the form \( t = ph \) (or \( s = ph, t = gh \)) for \( p, q \in \mathbb{Z} \).

We will consider a network of \( N \) nodes or agents, labeled by \( i \in V = \{1, \ldots, N\} \), interacting over a communication network. The topology of the network at time \( t \) will be represented by a directed graph \( \tilde{G}(t) = (V, E(t)) \) with an edge set \( E(t) \subset V \times V \). We consider that \((i, j) \in E(t)\) if and only if node \( i \) communicates to node \( j \) at time \( t \). The in-neighbors of node \( i \) at time \( t \) are denoted by \( \mathcal{N}_i(t) = \{j \in V : (i,j) \in E(t) \text{ and } j \neq i\} \). The matrix \( A(t) = [a_{ij}(t)] \in \mathbb{R}^{N \times N} \) represents the adjacency matrix of \( \tilde{G}(t) \) where \( a_{ij}(t) \neq 0 \) if edge \((i,j) \in E(t)\). Finally, \( 1 \in \mathbb{R}^N \) is the column vector whose entries are all ones.

At each time instant \( t \), every node synchronously measures the local continuous physical process \( r_i(t) : \mathbb{R} \rightarrow \mathbb{R} \), communicates with its neighbors and updates the state of its consensus algorithm. We ignore the delays induced by the communication and computation process. In the remainder of this paper, the sample \( r_i(t) \) is referred to as the reference signal (or input) of node \( i \) at time \( t \). Denote by \( \bar{r}(t) = \frac{1}{N} \sum_{i=1}^{N} r_i(t) \) the average of the reference inputs of the network at time \( t \).

Our objective is to design an nth-order dynamic average consensus algorithm that the nodes can utilize to asymptotically achieve the average of the reference inputs if the sum of relative deviation between the nth-order differences of any two reference inputs is bounded for some integer \( n \geq 1 \). We denote by \( \bar{x}_i(t) = (x_i^{[1]}(t), \ldots, x_i^{[n]}(t)) \in \mathbb{R}^n \) the consensus state of node \( i \) at time \( t \). The quantity of max \( \sup_{t \rightarrow \infty} |x_i^{[n]}(t) - \bar{r}(t - h)| \) is referred to as the steady-state error of nth-order dynamic average consensus algorithm. This can be interpreted as a measurement of how far the components of the consensus state \((x_i^{[1]}(t), \ldots, x_i^{[n]}(t))\) are from achieving the dynamic average consensus. Our algorithms will reach the dynamic average consensus with either a zero steady-state error or rendering the steady-state error smaller than (or equal to) any given bound.

3. First-order dynamic average consensus algorithm

In this section, we present a class of first-order algorithms to achieve the dynamic average consensus. The main references include Bertsekas and Tsitsiklis (1997), Blondel et al. (2005) and Olshesky and Tsitsiklis (2009). First of all, we define:

\[
M(t) = \max_{i \in V} x_i(t), \quad m(t) = \min_{i \in V} x_i(t),
\]

\[
D(t) = M(t) - m(t), \quad \Delta_r(t) = r_i(t) - r_j(t - h),
\]

\[
\Delta_{mx}(t) = \max_{i \in V} \Delta_r(t), \quad \Delta_{mn}(t) = \min_{i \in V} \Delta_r(t).
\]

By induction, the nth-order difference of \( r_i(t) \) is \( \Delta^{(n)} r_i(t) = \Delta^{(n-1)} r_i(t) - \Delta^{(n-1)} r_j(t - h) \) for \( n \geq 2 \) where \( \Delta^{(1)} r_i(t) = \Delta_r(t) \). We will use the notations \( \Delta^{(n)}_{mx}(t) = \max_{i \in V} \Delta^{(n)} r_i(t) \) and \( \Delta^{(n)}_{mn}(t) = \min_{i \in V} \Delta^{(n)} r_i(t) \) for \( n \geq 2 \).

In what follows, we will make use of the following assumption on \( \bar{r}(t) \) that was proposed in Jadbabaie and Lin (2003) and also used in Blondel et al. (2005) and Olshesky and Tsitsiklis (2009).

Assumption 3.1 (Periodical Strong Connectivity). There is some positive integer \( B \geq 1 \) such that, for all time instant \( t \geq 0 \), the directed graph \((V, \tilde{E}(t) \cup E(t + h) \cup \cdots \cup E(t + (B - 1)h))\) is strongly connected.

Assumption 3.2 (Relatively Bounded First-order Differences). For any \( h > 0 \), there exists a time invariant constant \( \theta > 0 \) such that

\[
\Delta R(t) := \Delta_{mx}(t) - \Delta_{mn}(t) \leq h \theta, \quad \forall t \geq 0.
\]

Remark 3.1. Inequality (1) becomes \( \max_{i \in V} \bar{r}_i(t) - \min_{i \in V} \bar{r}_i(t) \leq \theta h \) as \( h \rightarrow 0 \). Hence, Assumption 3.2 can be viewed as the discretized version of the property \( \max_{i \in V} \bar{f}_i(t) - \min_{i \in V} \bar{f}_i(t) \leq \theta \) for some fixed \( \theta \geq 0 \) and all time instant \( t \geq 0 \).

We propose the First-Order Dynamic Average Consensus algorithm (the FODAC algorithm for short) below to reach the dynamic average consensus:

\[
x_i(t + h) = x_i(t) + \sum_{j \neq i} a_{ij}(t) (x_j(t) - x_i(t)) + \Delta_r(t),
\]

when the reference input \( r(t) \) satisfies Assumption 3.2.
Remark 3.2. The FODAC algorithm can be rewritten as:

\[ \{ x_i(t) + h(t) - x_i(t) \} / h = \delta \sum_{j \neq i} a_{ij}(t)(x_j(t) - x_i(t)) + 9r_i(t - h - t) / h, \quad (3) \]

where the parameters \( \delta \) and \( h \) satisfy \( \delta h = 1 \). Observe that (3) is close to the discretized version of the continuous-time dynamic consensus algorithm in Spanos et al. (2005) but is not exactly the same. If \( h \to 0 \), then \( \delta = \frac{1}{h} \to \infty \), and thus the right-hand side of (3) is not well defined. ●

We will further suppose that the coefficients \( a_{ij}(t) \) in the FODAC algorithm satisfy the following two assumptions.

Assumption 3.3 (Blondel et al., 2005. Non-degeneracy). There exists a constant \( \alpha > 0 \) such that \( a_{ii}(t) = 1 - \sum_{j \neq i} a_{ij}(t) \geq \alpha \), and \( a_{ij}(t) \) \((i \neq j)\) satisfies \( a_{ij}(t) \in [0, 1] \), \( \forall t \geq 0 \).

Assumption 3.4 (Olfati-Saber & Murray, 2004. Balanced Communication). There hold that \( I(A(t)) = 1 \) and \( A(t) = 1 \), \( \forall t \geq 0 \).

Equivalently, the matrix \( A(t) \) is referred to as doubly stochastic, each of whose rows and columns sums to 1. Assumption 3.4 renders the conservation property \( \sum_{i=1}^{n} x_i(t + h) = \sum_{i=1}^{n} x_i(t) \) which is essential to reach the dynamic average consensus. It plays a similar role in achieving the static average consensus (Olfati-Saber & Murray, 2004).

We now proceed to analyze the FODAC algorithm. Let us fix \( k \in V \) for every \( s \) and define \( D_k = \{ k \} \). By Assumption 3.1, there is a non-empty set \( D_s \subset V \setminus \{ k \} \) of nodes such that for all \( s \in D_k \), node \( k \) communicates to node \( i \) at least once during the time frame \([s, s + (B - 1)h]\). By induction, a set \( D_{s+1} \subset V \setminus \{ D_s \} \) can be defined by considering those nodes \( j \) to which some \( s \in D_s \cup \ldots \cup D_{s+1} \) communicates at least once during the time frame \([s + \ell B, s + (\ell + 1)B - 1)h\]. By Assumption 3.1, \( D_{s+1} \neq \emptyset \) as long as \( V \setminus (D_0 \cup \ldots \cup D_s) \neq \emptyset \). Thus, there exists \( L \leq N - 1 \) such that the collection of \( D_0, \ldots, D_L \) is a partition of \( V \).

Lemma 3.1. Consider the FODAC algorithm and suppose that Assumptions 3.1 and 3.3 hold. Let \( s \geq 0 \) and \( k \in V \) be fixed and consider the associated \( D_s, \ldots, D_L \). Then for every \( \ell \in \{ 1, \ldots, L \} \), there exists a real number \( \eta_\ell > 0 \) such that for every integer \( p \in \{ LB, \ldots, LB + B - 1 \} \), and \( i \in D_\ell \), it holds that for \( t = s + ph \)

\[ x_i(t) \geq m(s) + \sum_{q=0}^{p-1} \Delta r_{\min}(s + qh) + \eta_\ell(x_k(s) - m(s)). \]

(4)

\[ x_i(t) \leq M(s) + \sum_{q=0}^{p-1} \Delta r_{\min}(s + qh) - \eta_\ell(M(s) - x_k(s)). \]

(5)

The following theorem is the main result in this section and shows the convergence properties of the FODAC algorithm.

Theorem 3.1. Let \( \delta_1 > 0 \) be a positive constant and \( h_1 = \frac{\delta_1 h}{2} \). Under Assumptions 3.1–3.4, the implementation of the FODAC algorithm with \( h \in (0, h_1] \) and initial conditions \( x_i(0) = r_i(-h), i \in \{ 1, \ldots, N \} \), achieves the dynamic average consensus with a nonzero steady-state error upper bounded by \( \delta_1 \).

The following corollary states an interesting special case of Theorem 3.1 when \( \lim_{\ell \to \infty} \Delta r_2(t) = 0 \) for any \( h > 0 \).

Corollary 3.1. Under Assumptions 3.1, 3.3, 3.4 and \( \lim_{\ell \to \infty} \Delta r_2(t) = 0 \) for any \( h > 0 \), the implementation of the FODAC algorithm with any \( h > 0 \) ensures that \( \lim_{\ell \to \infty} |x_i(t) - x_j(t)| = 0 \) for any \( i, j \in V \). Furthermore, if initial state \( x_i(0) = r_i(-h), i \in \{ 1, \ldots, N \} \), the dynamic average consensus is achieved with a zero steady-state error.

4. Higher-order algorithms for dynamic average consensus

In this section, we present \( n \)-th-order algorithms for dynamic average consensus where \( n \geq 2 \). First of all, let us consider the case of \( n = 2 \). We will assume that the reference inputs satisfy the following condition weaker than Assumption 3.2.

Assumption 4.1 (Relatively Bounded Second-order Differences). For any \( h > 0 \), there exists a time invariant constant \( \theta_2 > 0 \) such that

\[ \Delta(2)_{\max}(t) - \Delta(2)_{\min}(t) \leq \theta_2 h, \quad \ell \geq 0 \]

Correspondingly, we propose the following Second-Order Dynamic Average Consensus algorithm (the SODAC algorithm for short)

\[ x_i^{(2)}(t + h) = x_i^{(2)}(t) + \sum_{j \neq i} a_{ij}(t)(x_j^{(2)}(t) - x_i^{(2)}(t)) + x_i^{(1)}(t + h), \]

\[ x_i^{(1)}(t + h) = x_i^{(1)}(t) + \sum_{j \neq i} a_{ij}(t)(x_j^{(1)}(t) - x_i^{(1)}(t)) + \Delta(2) \]

and its convergence properties are described in the following theorem and corollary.

Theorem 4.1. Let \( \delta_2 > 0 \) be a positive constant and \( h_2 = \frac{\delta_2 h}{2} \). Under Assumptions 3.1, 3.3, 3.4 and 4.1, the implementation of the SODAC algorithm with \( h \in (0, h_2] \) and initial states \( x_i^{(1)}(0) = \Delta r_i(-h), x_i^{(2)}(0) = r_i(-h), i \in \{ 1, \ldots, N \}, \) achieves the dynamic average consensus with a nonzero steady-state error upper bounded by \( \delta_2 \).

Corollary 4.1. Under Assumptions 3.1, 3.3, 3.4 and \( \lim_{\ell \to \infty}(\Delta(2)_{\max}(t) - \Delta(2)_{\min}(t)) = 0 \) for any \( h > 0 \), the implementation of the SODAC algorithm with \( h > 0 \) and initial states \( x_i^{(1)}(0) = \Delta r_i(-h), x_i^{(2)}(0) = r_i(-h), i \in \{ 1, \ldots, N \} \), achieves the dynamic average consensus with a zero steady-state error.

Now, let us consider the following general \( n \)-th Order Dynamic Average Consensus algorithm (the NODAC algorithm for short).

\[ x_i^{(n)}(t + h) = x_i^{(n)}(t) + \sum_{j \neq i} a_{ij}(t)(x_j^{(n)}(t) - x_i^{(n)}(t)) + x_i^{(n-1)}(t + h), \]

\[ x_i^{(n-1)}(t + h) = x_i^{(n-1)}(t) + \sum_{j \neq i} a_{ij}(t)(x_j^{(n-1)}(t) - x_i^{(n-1)}(t)) + \Delta(n) r_i(t), \quad \ell \in \{ 2, \ldots, n \}. \]

Remark 4.1. In Ren, Moore, and Chen (2007), the authors propose a continuous-time higher-order consensus algorithm to allow higher-order derivatives converge to common values. While related, the problem statement of Ren et al. (2007) is different from ours. ●

The previous algorithm is the cascade of \( n \) FODAC algorithms and can be compactly rewritten in the following vector form

\[ x_i^{(2)}(t + h) = A(t)x_i^{(2)}(t) + x_i^{(n-1)}(t + h), \]

\[ x_i^{(1)}(t + h) = A(t)x_i^{(1)}(t) + \Delta(2) t(t), \quad \ell \in \{ 2, \ldots, n \}. \]

The above NODAC algorithm is able to track the average of reference inputs which satisfy the following condition under which Theorem 4.2 holds.

Assumption 4.2 (Relatively Bounded \( n \)-th-order Differences). For any \( h > 0 \), there exists a time invariant constant \( \theta_n > 0 \) such that

\[ \Delta(2)_{\max}(t) - \Delta(2)_{\min}(t) \leq \theta_n h, \quad \forall t \geq 0 \]

The following theorem and corollary show the convergence properties of \( n \)-th-order dynamic average consensus algorithm.
Theorem 4.2. Let \( \delta_n \) be a positive constant and \( h_n = \frac{\delta_n^{\frac{1}{2}}}{2^n - \theta_0(NB^{-1})} \).
Under Assumptions 3.1, 3.3, 3.4 and 4.2, the implementation of the NODAC algorithm with \( h \in \{0, h_n\} \) and initial states \( \chi_i(0) = \Delta^{(n-\ell)}r_i(-h) \) \( (\ell = 1, \ldots, n-1) \), \( \chi_i(n) = r_i(h) \) for all \( i \in \{1, \ldots, N\} \), achieves the dynamic average consensus with a nonzero steady-state error upper bounded by \( \delta_n \).

Proof. The case of \( n = 1 \) has been proven in Theorem 3.1. By using similar arguments in Theorem 4.1, we can finish the proof in an inductive way.

Corollary 4.2. Under Assumptions 3.1, 3.3, 3.4 and \( \lim_{n \to \infty} (\Delta^{(n)}r_{\text{max}}(t) - \Delta^{(n)}r_{\text{min}}(t)) = 0 \) with any \( h > 0 \), the implementation of the NODAC algorithm for any \( h > 0 \) and initial states \( \chi_i(0) = \Delta^{(n-\ell)}r_i(-h) \) \( (\ell = 1, \ldots, n-1) \), \( \chi_i(n) = r_i(h) \) for all \( i \in \{1, \ldots, N\} \), achieves the dynamic average consensus with a zero steady-state error.

5. Extensions
This section includes some remarks about the possible extension of the presented results.

5.1. Discussion on the choice of the order for the dynamic average consensus algorithm
If Assumption 4.2 holds, m-th-order dynamic consensus algorithm can reach the dynamic average consensus for any \( m > n \). However, we need a smaller \( h \) than the NODAC algorithm to render the steady-state error smaller than the given bound. Then there is no advantage to use m-th-order average dynamic consensus algorithm when Assumption 4.2 is satisfied.

5.2. Discussion on Assumption 4.2
It can be shown that for any \( m \)-th-order polynomial \( f(t) = \sum_{i=0}^{n} a_i t^i \), there holds that \( \Delta^{(m)}f(t) = a_n n! h \). Hence, any set of \( m \)-th-order polynomials satisfies Assumption 4.2 with \( h_{n+1} \) if \( h_n \).

If the reference inputs \( r_i(t) \) take the form of \( r_i(t) = v(t) + \tilde{r}_i(t) \), \( i \in V \), and the function \( \tilde{r}_i(t) \) is a linear combination of polynomials, the logarithmic function, periodic functions and other functions whose \( m \)-th-order differences are bounded, then Assumption 4.2 also holds for any common \( v(t) \) even when \( m \)-th-order difference of \( v(t) \) explodes, e.g., like the exponential function. It is worth mentioning that it is unnecessary for Assumption 4.2 to hold that \( \Delta^{(m)}r_i(t) \) be bounded for all \( i, t \geq 0 \).

5.3. Discussion on Assumption 3.1
In the case that the communication is symmetric; i.e., when \((i,j) \in E(t)\) if and only if \((j,i) \in E(t)\), then Assumption 3.1 in Corollary 4.2 can be weakened into:

Assumption 5.1 (Eventual Strong Connectivity). The directed graph \((V, \cup_{t \in T} E(t))\) is strongly connected for all time instant \( t \geq 0 \).

Corollary 5.1. Suppose \( g(t) \) is undirected. Under Assumptions 3.3, 3.4 and 5.1 and \( \lim_{n \to \infty} (\Delta^{(n)}r_{\text{max}}(t) - \Delta^{(n)}r_{\text{min}}(t)) = 0 \) with any \( h > 0 \), the implementation of the NODAC algorithm with \( h > 0 \) and initial states \( \chi_i(0) = \Delta^{(n-\ell)}r_i(-h) \) \( (\ell = 1, \ldots, n-1) \), \( \chi_i(n) = r_i(h) \) for all \( i \in \{1, \ldots, N\} \), achieves the dynamic average consensus with a zero steady-state error.

If the communication is symmetric, Assumption 3.1 in Corollary 4.2 can also be replaced with the assumption in Proposition 2 of Moreau (2005); i.e., for any time instant \( t \geq 0 \), there is a node connected to all other nodes in the undirected graph \((V, \cup_{t \in T} E(t))\). It is interesting to further think about the weaker assumption in Proposition 1 of Moreau (2005); i.e., there exists an integer \( B \geq 1 \) such that for any time instant \( t \geq 0 \), there is a node connected to all other nodes in the directed graph \((V, E(t) \cup E(t+h) \cup \cdots \cup E(t+(B-1)h))\).

5.4. The robustness to joining and leaving of nodes
If some nodes join the network at some time \( t_0 > 0 \) during the implementation of the NODAC algorithm, all the nodes in the new network are able to reach the new dynamic average consensus as long as the joining nodes choose their “initial” states at time \( t_0 \) according to the rules in Theorem 4.2 and Assumptions 3.1, 3.3, 3.4 and 4.2 are satisfied for the new network.

To make the NODAC algorithm adaptive to the departure of some nodes, we slightly modify its implementation. Assume node \( k \) wants to leave the network at some time \( t_k \) and \((k,i) \in E(t_k)\) for some node \( i \). Node \( k \) sends the value of \( x_k^{(0)}(t_k) - r_k(t_k) \) to node \( i \), and then node \( i \) updates its values according to the NODAC algorithm by replacing the top equation in the NODAC algorithm with the following:

\[
\chi_i^{(n)}(t_0+h) = \chi_i^{(n)}(t_0) + \sum_{j \neq i} a_{ij}(t_0)(\chi_j^{(n)}(t_0) - \chi_i^{(n)}(t_0))
+ \chi_i^{(n-1)}(t_0+h) + (\chi_i^{(n)}(t_0) - r_k(t_0)).
\]

All other remaining nodes update their values according to the NODAC algorithm at time \( t_0 \). After time \( t_0 \), the remaining nodes in the network update their values according to the NODAC algorithm, and then the dynamic average consensus is reached if Assumptions 3.1, 3.3, 3.4 and 4.2 are satisfied for the new network. The update law (6) ensures the conservation property of \( \sum_{j \neq k} \chi_j^{(n)}(t_0+h) = \sum_{j \neq k} r_j(t_0) \).

5.5. Asynchronous first-order dynamic average consensus algorithm
In this part, the asynchronism is incorporated into the FODAC algorithm. First, let us define the following notations: a set \( T_0 \) of time instants when node \( i \) measures \( r_i(t) \); a variable \( \tau_i(t) \) which denotes the latest time instant before time \( t \) in \( T_0 \). We adopt the partial asynchronism time model adapted from Bertsekas and Tsitsiklis (1997); i.e., there exists a positive integer \( B_0 \) such that \( t - B_0 h \leq \tau_i(t) \leq t \) for each \( i \in V \) and each \( t \geq 0 \).

Asynchronous First-Order Dynamic Average Consensus algorithm (asynchronous FODAC algorithm for short) is given by: if \( t \in T_0 \), node \( i \) measures \( r_i(t) \) and updates its value according to the following rule:

\[
x_i(t+h) = x_i(t) + \sum_{j \neq i} a_{ij}(t)(x_j(t) - x_i(t))
+ (r_i(t) - r_i(\tau_i(t)));
\]

otherwise, node \( i \) sets \( x_i(t+h) = x_i(t) \).

Assumption 5.2 (Bounded First-order Differences). For any \( h > 0 \), there exist time invariant constants \( p_1 > 0 \) and \( p_2 > 0 \) such that \( \Delta^{(1)}r_{\text{max}}(t) \leq h p_1 \) and \( \Delta^{(1)}r_{\text{min}}(t) \geq h p_2 \) hold for all \( t \geq 0 \).

Theorem 5.1. Let \( \delta_0 \) be a positive constant and \( h_1 = \frac{\delta_0}{4 h_1 (p_1 + p_2)(N+1)(B_0-1)} \).
Under Assumptions 3.1, 3.3, 3.4 and 5.2, the implementation of the asynchronous NODAC algorithm with the partial asynchronism time model, \( h \in (0, h_1) \) and initial conditions \( x_i(0) = r_i(-h), i \in \{1, \ldots, N\} \), achieves dynamic average consensus with a nonzero steady-state error upper bounded by \( \delta_1 \).
6. Simulations

In this section, we present several examples with their simulations to demonstrate the effectiveness of our theoretical results.

6.1. Example 1

We first illustrate the conclusion of Corollary 3.1 with a simulation. Let us consider a network consisting of four nodes, labeled 1 through 4. Suppose that the graph $\tilde{G}(t)$ satisfies Assumption 3.1 with $B = 4$. The reference inputs are given by:

$$r_1(t) = 5 \sin t + \frac{10}{t+2} + 1,$$

$$r_2(t) = 5 \sin t + \frac{10}{(t+2)^2} + 2,$$

$$r_3(t) = 5 \sin t + \frac{10}{(t+2)^3} + 3,$$

$$r_4(t) = 5 \sin t + 10e^{-t} + 4.$$

Fig. D.1 shows the tracking errors of the nodes asymptotically converge to zero.

6.2. Example 2

Now, we provide an example to illustrate the robustness of the NODAC algorithm. Consider a network with five nodes. The graph $\tilde{G}$ is fixed when no node joins or leaves the network. The reference inputs are given by:

$$r_1(t) = t + 1 + 5 \sin t,$$

$$r_2(t) = t - 1 + 5 \sin t,$$

$$r_3(t) = t + 5 \sin t,$$

$$r_4(t) = t - 50 + 5 \sin t.$$

It can be readily verified that Assumption 3.2 holds with $\theta = 0$. Thus we choose the NODAC algorithm with $h = 1$. During the simulation, node 5 leaves the network at time 50 and joins the network at time 100 again. Fig. D.2 provides the consensus states in comparison with the average of the reference inputs.

7. Conclusions

We have proposed a class of discrete-time dynamic average consensus algorithms and analyze their convergence properties. Due to slow convergence rates of the algorithms, tracking is shown at the expense of frequent communication and higher throughput. Future work will explore the algorithms application for sensor fusion.
where we are using the property of (A.3) in the last two inequalities. Applying repeatedly (A.4) we have that, for any integer \(p \in [0, (\mathcal{L}B + B - 1)]\), the following holds for \(t = ph\)

\[
x(t) - m(0) - \sum_{q=0}^{p-1} \Delta r_{\min}(q) \geq a^{p-1}(x(h) - m(0) - \Delta r_{\min}(0))
\]

\[
\geq a^p(x(0) - m(0)) \geq a^p (x_0(0) - m(0)).
\]

where \(a_0 = a^{NB-1}\) and we are using the properties of (A.3) and \(x_0(0) - m(0) \geq 0\). This proves inequality (4) for the nodes in \(\mathcal{D}_0 = [k]\) and for any integer \(p \in [0, (\mathcal{L}B + B - 1)]\).

Now we proceed by induction on \(\ell\). Suppose that (4) holds for some \(0 \leq \ell < L\); then we should show (4) for \(i \in \mathcal{D}_{\ell+1}\). It follows from the construction of the sets of \(\mathcal{D}_0, \ldots, \mathcal{D}_L\) that there exists some time \(t' \in [\mathcal{E}h, (\mathcal{L}B + B - 1)h]\) that has \(a_{\ell}(t') \neq 0\) for some \(j \in \mathcal{D}_0 \cup \cdots \cup \mathcal{D}_{\ell}\) and \(i \in \mathcal{D}_{\ell+1}\). By the induction hypothesis, we have that for all integers \(p \in [\mathcal{E}h, (\mathcal{L}B + B - 1)]\), there exists some \(\eta_i > 0\) such that the following holds for \(t = ph\)

\[
x(t) - m(0) - \sum_{q=0}^{p-1} \Delta r_{\min}(qh) \geq \eta_i (x_k(0) - m(0)).
\]

Consequently, as in (A.4), we have

\[
x(t') - m(0) - \sum_{q=0}^{\ell'-1} \Delta r_{\min}(qh) \geq a_{\ell'}(x_0(0) - m(0)).
\]

Following along the same lines as in (A.4), we have that

\[
x(t+h) - m(0) - \sum_{q=0}^{p} \Delta r_{\min}(qh) \geq \eta_{\ell+1}(x_k(0) - m(0)),
\]

holds for all \(p \in [(\ell+1)h, (\mathcal{L}B + B - 1)h]\), where \(\eta_{\ell+1} = \alpha^{(N-\mathcal{L}B)h_{\ell+1}}\) and \(t = ph\). This establishes (4) for \(i \in \mathcal{D}_{\ell+1}\). By induction, we have shown that (4) holds. The proof for (5) is analogous.

Appendix B. Proof of Theorem 3.1

Proof. Let \(\eta = \alpha^{\frac{1}{N}h_{(N+1)B-1}}\) then \(\eta \leq \eta_i\) for any \(\ell \in [1, \ldots, N-1]\). By replacing \(s\) and \(t\) in (4) with \(t\) and \(t_1 = t + (\mathcal{L}B + B - 1)h\) respectively, we have that for every \(t > 0\), there holds that

\[
m(t_1) = \min_{\ell \in [1, \ldots, \ell]} \min_{i \in \mathcal{D}_t} x_i(t_1)
\]

\[
\geq m(t) + \sum_{q=0}^{\ell-1} \Delta r_{\min}(qh) + \min_{i \in \mathcal{D}_t} x_i(t) - m(t)
\]

\[
\geq m(t) + \sum_{q=0}^{\ell-1} \Delta r_{\min}(qh) + \eta(x_k(t) - m(t)).
\]

Similarly, we have

\[
M(t_1) \leq M(t) + \sum_{q=0}^{\ell-1} \Delta r_{\max}(qh) - \eta(M(t) - x_k(t)).
\]

Combining the above two inequalities gives that

\[
D(t_1) \leq (1 - \eta)D(t) + \sum_{q=0}^{\ell-1} \Delta R(qh).
\]

Let us denote \(T_k = k(NB - 1)h\) for an integer \(k \geq 1\). From (A.2), we know that \(D(t + h) \leq D(t) + \Delta R(t)\). Thus we have

\[
D(t + T_k) \leq (1 - \eta)^k D(t) + \sum_{q=0}^{\ell-1} \Delta R(qh)
\]

and thus,

\[
D(T_k) \leq (1 - \eta)^k D(0) + \Omega(n),
\]

where

\[
\Omega(n) = (1 - \eta)^{T_k} \sum_{q=0}^{\ell-1} \Delta R(qh) + \cdots + \sum_{q=0}^{T_k-1} \Delta R(qh).
\]

For any \(t \geq 0\), let \(\ell_t\) to be the largest integer such that \(\ell_t(NB - 1)h < t\), and \(\ell_t := \Omega(\ell_t) + \sum_{q=0}^{\ell_t-1} \Delta R(qh)\). Thus for all \(t \geq 0\) it follows that

\[
D(t) \leq D(T_{\ell_t}) + \sum_{q=0}^{\ell_t-1} \Delta R(qh)
\]

\[
\leq (1 - \eta)^{T_{\ell_t}} D(0) + \tilde{\Omega}(t)
\]

\[
\leq (1 - \eta)^{T_{\ell_t}} D(0) + \tilde{\Omega}(t).
\]

Since \(\Delta R(t) \leq h\theta D(t)\) is input-to-output stable with ultimate bound \(\Sigma \leq 4\theta h NB - 1)^{\frac{1}{2}} \leq 4\theta h (NB - 1)\alpha^{-\frac{1}{2}N(N+1)B+1}\); i.e., there exist \(\gamma > 0\) and \(0 < \lambda < 1\) such that

\[
D(t) \leq \max\{\gamma \lambda^t, \Sigma\}, \quad \forall t \geq 0.
\]

Choose as initial state \(x(0) = r_i(-h)\) for all \(i \in \{1, \ldots, N\}\). By Assumption 3.4, the following conservation property of the FODAC algorithm is satisfied for all \(t \geq 0\):

\[
\sum_{i=1}^{N} x_i(t + h) = \sum_{i=1}^{N} x_i(t) + \sum_{i=1}^{N} \Delta r_i(t)
\]

\[
= \sum_{i=1}^{N} x_i(0) + \sum_{i=1}^{N} \sum_{q=0}^{\ell_i} \Delta r_i(qh)
\]

\[
= \sum_{i=1}^{N} x_i(0) + \sum_{i=1}^{N} (r_i(t) - h_i(-h)) = \sum_{i=1}^{N} r_i(t),
\]

where we have used the induction in Line 2 of the above expressions.

It follows from (B.3) that \(m(t + h) \leq \frac{1}{N} \sum_{i=1}^{N} r_i(t) \leq M(t + h)\) and thus

\[
\max \lim_{t \to \infty} \left| x_i(t) - \frac{1}{N} \sum_{i=1}^{N} r_i(t - h) \right| \leq \limsup_{t \to \infty} D(t) \leq \Sigma.
\]

Hence, for any given \(\delta_i > 0\), choosing \(h \leq h_i\) gives an steady-state error \(\Sigma \leq \delta_i\). In other words, choosing a step of size \(h\) induces at least an error of order \(4\theta h (NB - 1)\alpha^{-\frac{1}{2}N(N+1)B+1}\).

Appendix C. Proof of Theorem 4.1

Proof. Note that the algorithm for \(x^{[1]}(t)\) in the SODAC algorithm has the same form as the FODAC algorithm, and can be obtained
from this by replacing $\Delta r(t)$ with $\Delta x^{(2)} r(t)$. Since Assumption 4.1 holds, it follows from Theorem 3.1 that by choosing the initial state as $x^{(1)}(0) = \Delta r(t)$ we can find $\Gamma > 0$ and $0 < \lambda < 1$ such that for all $t \geq 0$ and all $i \in \{1, \ldots, N\}$, there holds that

$$|x_i^{(1)}(t) - \frac{1}{N} \sum_{j=1}^{N} \Delta r_i(t - h)| \leq D_i^{(1)}(t) \leq \max\{\Gamma r_i^{\frac{\lambda}{1-\lambda}}, \varepsilon_i\},$$

where $D_i^{(1)}(t) = \max_{v \in V} x_i^{(1)}(t) - \min_{v \in V} x_i^{(1)}(t)$ and $\varepsilon_i \leq 4h\rho_i^2(N - 1)\alpha^{-\frac{\lambda(N+1)}{N+1+\beta+1}}$

Hence, there exists a finite $\tilde{t} > 0$ such that $\Gamma r_i^{\frac{\lambda}{1-\lambda}} \leq \varepsilon_i$ for all $t \geq \tilde{t}$. In this way, $x_i^{(2)}(t)$ in the SODAC algorithm can be written in the following way for $t \geq \tilde{t}$

$$x_i^{(2)}(t + h) = x_i^{(2)}(t) + \sum_{j=1}^{N} a_j(t)(x_j^{(2)}(t) - x_i^{(2)}(t)) + d_i(t),$$

with a reference input $d_i(t) = \frac{1}{N} \sum_{j=1}^{N} \Delta r_i(t) + \theta_i(t)$ and $|\theta_i(t)| \leq \varepsilon_i$. Note that for all $t \geq \tilde{t}$, there holds that

$$\max_{v \in V} d_i(t) - \min_{v \in V} d_i(t) \leq 2\varepsilon_i$$

Hence, (C.1) has the same form as the FODAC algorithm, and can be obtained from it by replacing $\Delta r_i(t)$ with $d_i(t)$, where $\theta = 8h\rho_i^2(N - 1)\alpha^{-\frac{\lambda(N+1)}{N+1+\beta+1}}$ in Assumption 3.2.

Furthermore, consider as initial states $x^{(2)}(0) = r_i(-h)$ for all $i \in \{1, \ldots, N\}$, Similarly to (B.3) with $\Delta r_i(t)$ instead of $r_i(t)$, we can obtain the following conservation property of the SODAC algorithm for every $t \geq 0$

$$\sum_{i=1}^{N} x_i^{(1)}(0) + t = \sum_{i=1}^{N} \Delta r_i(t), \quad \sum_{i=1}^{N} x_i^{(2)}(0) + t = \sum_{i=1}^{N} r_i(t)$$

By using similar arguments to those employed in Theorem 3.1, we have that there exist $\Gamma > 0$ and $0 < \lambda < 1$ such that for all $t \geq \tilde{t}$ and all $i \in V$, there holds

$$|x_i^{(2)}(t) - \frac{1}{N} \sum_{j=1}^{N} r(t - h)| \leq D_i^{(2)}(t) \leq \max\{\Gamma r_i^{\frac{\lambda}{1-\lambda}}, \varepsilon_i\},$$

where $D_i^{(2)}(t) = \max_{v \in V} x_i^{(2)}(t) - \min_{v \in V} x_i^{(2)}(t)$ and $\varepsilon_i = 4h\theta_i^2(N - 1)\alpha^{-\frac{\lambda(N+1)}{N+1+\beta+1}}$.

For any given $\delta_2 > 0$, choosing $h \leq h_2$ leads to $\varepsilon_i \leq \delta_2$.

**Appendix D. The sketch of the proof for Theorem 5.1**

Following the same lines in Lemma 3.1 and Theorem 3.1, we utilize Assumption 5.2 to have that there exist $\Gamma > 0$ and $\lambda > 0$ such that

$$D(t) \leq \max\{\Gamma r_i^{\frac{\lambda}{1-\lambda}}, \varepsilon_i\}, \quad \forall t \geq 0 \quad (D.1)$$

where $\varepsilon_i \leq 4B_i h(\rho_i - \rho_2)(NB - 1)^{\frac{1}{\theta}} \leq 4B_i h(\rho_i - \rho_2)(NB - 1)\alpha^{-\frac{\lambda(N+1)}{N+1+\beta+1}}$

Since $x_i(0) = r_i(-h)$ for all $i \in \{1, \ldots, N\}$, the conservation property of $\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} r_i(t)$ holds.

From Assumption 5.2 and the property of $t - r_i(t) < B_i h$, it can be shown that

$$\frac{1}{N} \sum_{i=1}^{N} r(t - h) - (B_i - 1)h \rho_i \leq \frac{1}{N} \sum_{i=1}^{N} x_i(t)$$

Combining (D.1) and (D.2) gives the following estimate for the steady-state error

$$\max_{i \in V} \limsup_{t \to \infty} |x_i(t) - \frac{1}{N} \sum_{i=1}^{N} r(t - h)| \leq 4B_i h(\rho_i - \rho_2)(NB - 1)\alpha^{-\frac{\lambda(N+1)}{N+1+\beta+1}} + \max\{\rho_i, |\rho_i|\}(B_i - 1)h.$$


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