Distributed robust frequency regulation of smart power grid with renewable integration

Hunmin Kim and Minghui Zhu

Abstract—This paper investigates distributed frequency regulation of multi-machine power systems with renewable integration. The key challenge is to completely compensate the uncertainties induced by renewable energy. We integrate different grid components to achieve robust frequency regulation; the battery is used to filter out high-frequency components of renewable energy, demand response is utilized to deal with their medium-frequency components and conventional synchronous generators are exploited to handle their low-frequency components. The proposed distributed controllers are based internal model principle and its induced stability is formally analyzed.

I. INTRODUCTION

A. Motivation

Beyond current power grid, modernized power grid named smart grid is emerging. One important feature of smart grid is to integrate information and communication technologies relating power consumption and storage automatically [2]. Integrating new information to conventional generators such as demand response enables us to control in more flexible and efficient manner. Another feature is to merge variable renewable energy to the conventional grid which is emerging industries due to its profits [16].

It is necessary to prepare integration of these resources to current power grid since this poses possible challenges. Renewable energy can not be fully predicted, even more it is fluctuating fast and largely [7]. For example, solar photovoltaics generation can change up to 5 percent fifteen times in one minute [8]. When renewable energy is connected to the grid, these riddles have negative effect on grid stability. Unpredictability acts as disturbance to the system; large disturbance results in large fluctuation. Conventional generator cannot tracks high frequency fluctuation as it has large inertia.

B. Contribution

We studied distributed frequency regulation of smart grid integrating wind energy and market pricing control canceling disturbance induced by wind energy.

We first split wind power generation signal into three parts; low, medium, and high frequency parts. Each control authority adopts battery system which acts as low pass filter to cope with high frequency uncertainties. Then, we have control authority adopt local internal model to reconstruct uncertain low-medium signal asymptotically. Moreover, two different controllers are assigned to compensate different range of renewable power generation. Eventually, two controllers achieve global exponential frequency regulation via the network small gain theorem. Controller design method and analysis are presented at the end of the paper.

C. Literature review

Recently, many attempts addressing unpredictability of renewable energy generation have been done: several optimization to solve integration and energy congestion problem [5], adopting battery system for enabling integration of solar power generator [6], etc. However, few researches deal with integrating both renewable energy and demand [9].

Distributed control method has been studied extensively. The classic distributed control includes power system stabilizer (PSS) and automatic generation control (AGC) [23]. Recently, distributed control and decision making have been applied to solve a number of emerging issues in the smart grid: microgrid control [18], PHEV charging [17] and demand response [4], etc.

For distributed robust control in power grid, research [20] proposed decentralized hierarchical control architecture for fast frequency stability without requiring secondary control. Research [10] used distributed hierarchical approach for wide-area control adopting four generators and one synchronous condenser for new power system stabilizer. Microgrid research [21] applies internal model voltage controller and droop-based power-sharing controller to three-phase DG interface to increase disturbance rejection rate. However, most of papers are mainly focusing on disturbance attenuation, conventional purpose. The disturbance induced by renewable energy is different from conventional disturbance in the sense of its scale. For reliable stability, complete disturbance rejection is preferred under existence of large disturbance.

D. Notation

$\mathbb{R}$ denotes set of all real number and $\mathbb{R}_\geq 0$ denotes set of all positive real numbers and zero. $0_{1 \times n}$ implies a $1 \times n$ vector having all 0 elements. $n_c$ denotes elements $n_c \in \{1, \cdots, c\}$. $\|u\|_{t_1, t_2} = \sup_{t_1 \leq t \leq t_2} \|u(t)\|$. Matrix $I$ denotes identity matrix with proper dimension. $\text{diag}(A_1, \cdots, A_n)$ denotes a block matrix having $A_1 \text{ tp } A_n$ as main diagonal block and the off-diagonal blocks are all zero matrices.
II. SYSTEM MODEL

We introduce system parameters in section II-A and system model in section II-B.

A. System Parameters

Following parameters are used in the rest of the paper.

<table>
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<th>TABLE I: Generator variables</th>
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<th>TABLE III: Variables and parameters of wind power</th>
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B. Power system model

Power network is described by a graph $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V} \equiv \{1, \cdots, N\}$ denotes the set of buses in the network and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of transmission lines. $N$ is total number of buses. $\mathcal{N}_i$ denotes time invariant set of neighboring bus of $i \in \mathcal{V}$: $\mathcal{N}_i \equiv \{j \in \mathcal{V}\backslash \{i\}|(i,j) \in \mathcal{E}\}$. $(i,j) \in \mathcal{E}$ indicates power flow line from bus $i \in \mathcal{V}$ to $j \in \mathcal{N}_i$. We assume power network is undirected, that is $(i,j) \in \mathcal{E}$ if and only if $(j,i) \in \mathcal{E}$.

Each bus $i \in \mathcal{V}$ has local control authority and power sources/load: mechanical generation $P_{M,i}$, wind power generation $P_{ren,i}$, power demand $P_L$, and tie line connection $P_{ij}$ where $P_{ij} = -P_{ji}$. Positive tie line connection $P_{ij} > 0$ indicates positive power flow from $i \in \mathcal{V}$ to $j \in \mathcal{N}_i$.

1) Generator dynamics: The generator model is adopted from [23] with integrating wind power generation $P_{ren,i}$ [24]. Consider the model of control authority $i \in \mathcal{V}$ described by

$$
\frac{d\theta_i}{dt} = 2\pi w_i
$$

$$
\frac{dw_i}{dt} = -\frac{1}{m_i} \left( D_i w_i + \sum_{j \in \mathcal{N}_i} P_{ij} - P_{M,i} + P_{L,i} - P_{ren,i} \right)
$$

$$
\frac{dP_{M,i}}{dt} = -\frac{1}{T_{CH,i}} (P_{M,i} - P_{w,i})
$$

$$
\frac{dP_{L,i}}{dt} = -\frac{1}{T_{G,i}} \left( P_{w,i} + 1 \frac{R_i}{\pi} w_i - P_{ref,i} \right)
$$

and

$$
P_{ij}(t) = t_{ij}(\theta_i(t) - \theta_j(t))
$$

where $\theta_i$ and $\theta_j \in [0, 2\pi)$ for simplicity. First equation in (1) is called swing dynamics, denoting frequency fluctuation due to power imbalance. In general, we want frequency $w_i$ to be constant. However if there is any imbalance between power generation and demand, $w_i$ evolve. Frequency $w_i$ increases as total power generation surpasses power demand $P_{M,i} + P_{ren,i} > \sum_{j \in \mathcal{N}_i} P_{ij} + P_{L,i}$ and decreases in reverse case.

Third and forth equation in (1) denote turbine governor dynamics with reference input $P_{ref,i}$. Governor regulate torque of the rotor and eventually regulate frequency $w_i$ of the current to desired set point $w_i^*$. $R_i$ is a feedback loop gain which determines speed droop of characteristic. Equation (2) with second equation in (1) stands for the power flow between control authorities $i \in \mathcal{V}$ and $j \in \mathcal{N}_i$.

2) Wind power model: Consider the wind power generation system of control authority $i \in \mathcal{V}$ described by [19]:

$$
P_{w,i}(t) = \frac{1}{2} \rho \pi r_i^2 v_W^3(t)
$$

where $C_p$ is turbine’s power coefficient. It is hard to predict exact wind velocity $v_W(t)$ due to fluctuation. For this reason, we cannot predict exact nominal wind power generation $P_{ren,i}$, which makes it hard to compensate power imbalance caused by fluctuation of wind power. To deal with this problem, let us first divide wind power generation as three parts according to the previous research [22]:

$$
P_{ren,i}^H(t) = P_{ren,i}^H(t) + P_{ren,i}^M(t) + P_{ren,i}^L(t)
$$

where $P_{ren,i}^H(t)$, $P_{ren,i}^M(t)$, and $P_{ren,i}^L(t)$ denote high, medium, and low frequency parts of wind power generation.

To suppress high frequency wind power generation $P_{ren,i}^H(t)$, we adopt battery system which is set between wind turbine generator and conventional generator (1). The battery system acts as low pass filter [11] so that high frequency power be approximated as $P_{ren,i}^H(t) \simeq 0$, thus eliminating unpredictability of $P_{ren,i}^H(t)$. Consider transfer function of the battery system

$$
P_{ren, i}(s) = \frac{1}{(1+sT)} P_{ren, i}(s)
$$

where $P_{ren, i}(s)$ and $P_{ren, i}(s)$ denote input and output power of battery system, respectively. The effectiveness of transfer
function (5) is determined by the time constant $T$. We need a proper constant power $T$ which can sufficiently suppress high frequency wind power generation $P_{ren}^{W}$, and effect of high frequency wind $v_{i}^{W}(t)$. With the battery system, (4) becomes

$$P_{ren}(t) = P_{ren}^{M}(t) + P_{ren}^{L}(t).$$

We can approximate medium and low-frequency part, $0.0007 - 5$ cycles/h, of wind speed $v_{i}^{W}$ in (3) as Van der Hovens model which consists a combination of the sinusoidal function with a step function [22]. Medium and high frequency wind power generation $P_{ren}^{M}$ and $P_{ren}^{L}$ in (3) can also be approximated as a combination of sinusoidal functions with a step function since a higher order sinusoidal function can be expressed as a combination of lower order ones; for example, relation $\sin^{3}(x) = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)$ is valid. We will assume that medium and low frequency wind power generation is generated by partially known exosystem; each control authority knows structure of the exosystem but not state and initial state of it. Exosystem is introduced in the next subsection in detail.

3) Exosystem: In this section, we introduce exosystem for low and medium frequency wind power generation. Consider exosystem described by

$$\dot{v}_{i}(t) = \Omega_{i}v_{i}(t), \quad P_{ren,i}(t) = \Xi_{i}^{T}v_{i}(t) \tag{6}$$

where

$$\Omega_{i} = \text{diag}\left[\begin{array}{cc} 0 & -\rho_{i} \\ \rho_{i} & 0 \end{array}\right], \quad \ell \in \{1, \cdots, \ell_{i}\}$$

where $\ell_{i}$ denotes dimension of exosystem. $\Omega_{i} \in \mathbb{R}^{2\ell_{i}\times 2\ell_{i}}$, $\Xi_{i} \in \mathbb{R}^{2\ell_{i}\times 1}$ and $v_{i} \in \mathbb{R}^{2\ell_{i}}$. State $v_{i}(t)$ denotes exogenous signal representing the low and medium frequency wind power generation. Note that $P_{ren,i}$ in the exosystem (6) is a linear combination of sinusoidal functions having frequency $\rho_{i}$. The control authority $i$ cannot measure state $\dot{v}_{i}(t)$, initial state $v_{i}(t_{0})$, amplitude of phase. However, $\Omega_{i}$ and $\Xi_{i}$ are known to the control authority $i \in \mathcal{V}$, that is, frequencies and step function are known.

4) Demand model: Power demand $P_{Li}$ of control authority $i \in \mathcal{V}$ includes all local power consumption of consumer, instantaneous and aggregate demands. Demand can be separated as two parts [1]: $P_{Li} = P_{Li}^{E} + P_{Li}^{IE}$ where $P_{Li}^{E}$ and $P_{Li}^{IE}$ denote elastic and inelastic demand of control authority $i \in \mathcal{V}$ respectively. Elastic demand $P_{Li}^{E}$ is governed by following dynamics under assumption that marginal benefit function $b_{i}' + c_{i}'P_{Li}^{E}$ of consumer is a negatively proportional to power consumption:

$$\dot{P}_{Li}^{E} = b_{i} + c_{i}P_{Li}^{E} - \lambda_{i}(t) \tag{7}$$

where

$$b_{i} = \frac{b_{i}'}{\tau_{i}}, \quad c_{i} = \frac{c_{i}'}{\tau_{i}}, \quad \lambda_{i}(t) = \frac{\lambda_{i}'(t)}{\tau_{i}}.$$ 

$b_{i}'$ and $c_{i}'$ are power consumer benefit parameters, $\lambda_{i}'(t)$ is price of power, and $\tau_{i}$ is consumer response constant. Demand model (7) stands for the fact that load increases as marginal benefit is larger than price $\lambda_{i}(t)$. The way to choose parameters $b_{i}'$ and $c_{i}'$, and constant $\tau_{i}$ is introduced in [1]. We assume inelastic power demand $P_{Li}^{IE}$ is fully known to each control authority.

5) Compact model: Let $x_{i} \triangleq [\theta_{i} \ w_{i} \ P_{Mi} \ P_{vi} \ P_{Li}^{IE}]^{T}$ be the system state, $u_{i} \triangleq [\ref{P}_{i} \ \lambda_{i}]^{T}$ be the input of control authority $i$. Then generator model (1) and (2), and load demand model (7) are described compactly

$$\dot{x}_{i}(t) = A_{i}x_{i}(t) + B_{i}u_{i}(t) + C_{i}P_{ren} + \sum_{j \in \mathcal{N}_{i}} D_{ij}x_{j}(t) + E_{i}P_{Li}^{IE} + F_{i} \tag{8}$$

where each matrix can be found in (1) and (7). In (8), we assume that the system state $x_{i}(t)$, input $u_{i}(t)$, and inelastic load $P_{Li}^{IE}$ are measurable but wind power $P_{ren}(t)$ is not measurable and generated by partially known exosystem (6).

III. DISTRIBUTED ROBUST FREQUENCY REGULATION

In III-A, we state the frequency regulation problem this paper deals with. In III-B internal model principle is introduced to cope with uncertainty of wind power generation $P_{ren}$. In the rest of subsections, we study ISS of the local system. To this end, stability of the network is achieved.

A. Problem statement

Our regulation objective is to regulate the system state $x_{i}$ and input $u_{i}$ asymptotically converge to a manifold set $x_{i}^{*}$ for $i \in \mathcal{V}$. The manifold set $x_{i}^{*}(t) \triangleq [\theta_{i}^{*} \ w_{i}^{*} \ P_{Mi}^{*} \ P_{vi}^{*} \ P_{Li}^{IE}]^{T}$ and $u_{i}^{*}(t) \triangleq [\ref{P}_{i}^{*} \ \lambda_{i}]^{T}$ denote desired system state and input. We expect the system has following property on the manifold:

$$w_{i}^{*} = w^{*}, \quad \theta_{i}^{*}(t) = 2\pi t + \theta_{i}^{*}.$$ 

Each control authority $i \in \mathcal{V}$ has a common fixed frequency $w^{*}$, in general 60Hz. Phase shift $\theta_{i}^{*}$ may differ from each other, indicating non-zero $P_{ij}^{s}$ on the manifold. To achieve this goal, $P_{Mi}$ are expected to compensate low frequency wind power generation $P_{Li}^{IE}$ via input $P_{vi}^{s}$ and $P_{Li}^{IE}$ are expected to compensate medium frequency wind power generation $P_{Li}^{IE}$ via pricing input $\lambda_{i}$. Moreover two controllers are expected to achieve frequency stability of the system. Then the system consisting of (1), (2) and (7) has following manifold set for $i \in \mathcal{V}$:

$$\theta_{i}^{*}(t) = 2\pi w^{*}t + \theta_{i}^{*},$$

$$w_{i}^{*}(t) = w^{*},$$

$$P_{Mi}^{*}(t) = P_{Li}^{IE}(t) - P_{ren}^{L}(t) + D_{i}w^{*} + \sum_{j \in \mathcal{N}_{i}} P_{ij}^{s},$$

$$P_{vi}^{*}(t) = T_{Ch}(\dot{P}_{Li}^{IE}(t) - \dot{P}_{ren}^{L}(t)) + P_{Li}^{IE}(t) - P_{ren}^{L}(t) + D_{i}w^{*} + \sum_{j \in \mathcal{N}_{i}} P_{ij}^{s},$$

$$P_{ref_{i}}^{*}(t) = T_{G_{i}}T_{Ch}[(\dot{P}_{Li}^{IE}(t) - \dot{P}_{ren}^{L}(t)) + (T_{G_{i}} + T_{Ch})](\dot{P}_{Li}^{IE}(t) - \dot{P}_{ren}^{L}(t)) + P_{Li}^{IE}(t) - P_{ren}^{L}(t) + (D_{i} + \frac{1}{R})w^{*} + \sum_{j \in \mathcal{N}_{i}} P_{ij}^{s},$$

$$P_{Li}^{IE,*}(t) = P_{ren}^{M}(t).$$
Load demand is described by $P(t) = P_{\text{ren}}^L(t) + b_i$ for each control authority, $i \in \mathcal{V}$. Note that power flow $P_{ij}^r$ is constant and frequency evolution $\dot{w} = 0$ on the manifold because desired frequency $w^*$ is identical for each control authority, $w_i^* = w_j^*$ for all $i, j \in \mathcal{V}$. To ensure existence of the manifold, let us have the following assumptions.

**Assumption 3.1:** Inelastic power demand $P_{\text{IE}}^L(t)$ is a sufficiently smooth function for $\forall i \in \mathcal{V}$.

**Assumption 3.2:** For the set of desired power flow $P_{ij}^*$ for $\forall (i, j) \in \mathcal{E}$, there exists a set of solution $\theta_i^*$ and $\theta_j^*$ for equation $P_{ij}^* = \dot{t}_{ij}(\theta_i^* - \theta_j^*)$.

Our goal is to make system state and input converge manifold $x_i(t) \rightarrow x_i^*(t)$ and $u_i(t) \rightarrow u_i^*(t)$ asymptotically as $t \rightarrow \infty$. However, it is challenged by the fact that exosystem $v_j \in \mathbb{R}^{2\xi_j}$ and wind power generation $P_{\text{ren}}^L$, and $P_{\text{ren}}^M$ cannot be directly measurable, hence manifold (9) neither. To address this, we adopt internal model principle to asymptotically reconstruct (12) via state and input measurement.

**B. Local internal models**

We introduce local internal model in this subsection. For the notational simplicity, let us first define

\[
Q_{M_i}^*(t) \triangleq P_{\text{IE}}^L(t),
Q_{\text{ref}_i}^*(t) \triangleq T_{CH}P_{\text{IE}}^L(t) + P_{\text{IE}}^L(t),
\]

for control authority $i \in \mathcal{V}$. Let us combine 3rd and 6th equation in (9), separating known terms on the left and unknown terms on the right. Since unknown terms are produced by exosystem,

\[
\begin{align*}
-\dot{P}_{M_i}^*(t) + P_{ij}^E(t) + Q_{M_i}^*(t) + \sum_{j \in \mathcal{N}_i} P_{ij}^* + D_i w^* &= P_{\text{ren}}(t) + P_{\text{ren}}^M(t) \\
&= \Psi_i[v_i,1 \ v_i,2 \ v_i,1]T
\end{align*}
\]

for a row vectors $\Psi_i \in \mathbb{R}^{2\xi_i}$. Let us assume that $v_i,1$ is in ascending order with respect to corresponding frequency $\rho_i$ without loss of generality. $\Psi_i$ can be calculated from (6):

\[
\Psi_i = \begin{bmatrix} \star & 0 & \cdots & \star & 0 \end{bmatrix}
\]

where $\star$ means non-zero value. Moreover,

\[
\begin{bmatrix}
\check{v}_{i,1,r-1} \\
\check{v}_{i,2,r-1}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\check{P}_{i,r-1} & 0
\end{bmatrix}
\begin{bmatrix}
v_{i,1,r-1} \\
v_{i,2,r-1}
\end{bmatrix}, \quad r \in \{1, \ldots, \xi_i\}.
\]

where $\Psi_i \triangleq \text{diag}(\Phi_i, \Phi_{i,2})$ for $\forall r \in \{1, \ldots, \xi_i\}$. Now we have following assumption for the solvability of Sylvester equation.

**Assumption 3.3:** The pair $(\Psi_i, \Phi_i)$ is observable for $\forall i \in \mathcal{V}$.

Choose a controllable pair $(M_i, N_i)$ with $M_i$ Hurwitz for $i \in \mathcal{V}$. Then, there exists unique non-singular matrix $T_i$ for Sylvester equation (3):

\[
T_i \Phi_i - M_i T_i = N_i \Psi_i,
\]

Let $\dot{\vartheta}_i \triangleq T_i[v_i,1 \ v_i,2 \ v_i,1]$. Then, (10) becomes

\[
\Psi_i T_i^{-1} \dot{\vartheta}_i(t) = -P_{M_i}^*(t) + P_{\text{IE}}^E(t) + Q_{M_i}^*(t) + \sum_{j \in \mathcal{N}_i} P_{ij}^* + D_i w^*.
\]

Now we have local internal model candidate.

\[
\eta_i = M_i \dot{\vartheta}_i + N_i g_i(x_i, u_i, t)
\]

where

\[
M_i \dot{\vartheta}_i + N_i g_i(x_i, u_i, t) = -P_{M_i}^*(t) + P_{\text{IE}}^E(t) + Q_{M_i}^*(t) + \sum_{j \in \mathcal{N}_i} P_{ij}^* + D_i w^*.
\]

However, we want to separate internal model $\eta_i$ into two terms so that we can estimate $P_{\text{ren}}^L$, and $P_{\text{ren}}^M$ separately. This is always possible since $P_{\text{ren}}^L$, and $P_{\text{ren}}^M$ have different frequencies $\rho_{i,1}$, which means they do not share state $v_{i,1}$.

First, let us split dimension of internal model $\xi_i = \xi_i^L + \xi_i^M$ where $\xi_i^L$ and $\xi_i^M$ are low and medium frequency dimension respectively. Consider, $\Psi_i = [\Psi_i^L \ \Psi_i^M]$ where $\Psi_i^L \in \ell_{\xi_i^L}^T$ and $\Psi_i^M \in \ell_{\xi_i^M}^T$. Let us append $\ell_{\xi_i^L}$ zeros $0^{1 \times \xi_i^M}$ in both of $\Psi_i^L$ and $\ell_{\xi_i^M}$ zeros $0^{1 \times \xi_i^L}$ in front of $\Psi_i^M$ and define $\Psi_{i,1} \triangleq [\Psi_{i,1}^L \ 0^{1 \times \xi_i^M}]$ and $\Psi_{i,2} \triangleq [0^{1 \times \xi_i^L} \ \Psi_{i,2}^M]$. Then we have $\Psi_i = \Psi_{i,1} + \Psi_{i,2}$. Note that $P_{\text{ren}}^L = \Psi_{i,1} T_i^{-1} \dot{\vartheta}_i$ and $P_{\text{ren}}^M = \Psi_{i,2} T_i^{-1} \dot{\vartheta}_i$.

Measurable $\eta_i$ is expected to track unmeasurable $\dot{\vartheta}_i$ asymptotically in the internal model (12), the other words internal model can estimate uncertain wind power generation $P_{\text{ren}}^L$, and $P_{\text{ren}}^M$. If this is the case, each control authority can track uncertainty by adopting local internal model.

**C. Coordinate transformation one**

The purpose of first coordinate transformation is to transform the frequency regulation problem of (8) into stability problem of the manifold (15). This can be done by subtracting manifold (9) from the state and input variables.

Since $\eta_i(t)$ will follow $\dot{\vartheta}_i(t)$ asymptotically for $\forall i \in \mathcal{V}$, the estimation errors $|\Psi_{i,1} T_i^{-1} \dot{\vartheta}_i(t) - \Psi_{i,1} T_i^{-1} \dot{\vartheta}_i(t)|$ for $j \in \{1, 2\}$ converge to zero asymptotically. Therefore, with internal model candidate (12), it is plausible to use $\Psi_{i,1} T_i^{-1} \dot{\vartheta}_i(t)$ instead of $\Psi_{i,2} T_i^{-1} \dot{\vartheta}_i(t)$ in manifold (9) when subtracting manifold. Consider coordinate transformation described by

\[
\begin{align*}
\dot{\theta}_i(t) &\triangleq \theta_i(t) - (2\pi w^* t + \theta_i^*) \\
\check{w}_i(t) &\triangleq w_i(t) - w^* \\
\check{P}_{M_i}(t) &\triangleq P_{M_i}(t) + \Psi_{i,1} T_i^{-1} \dot{\vartheta}_i(t) - Q_{M_i}^*(t) \\
&- \sum_{j \in \mathcal{N}_i} P_{ij}^* - D_i w^* \\
\check{P}_{v_i}(t) &\triangleq P_{v_i}(t) + T_{CH} \Psi_{i,1} T_i^{-1} \dot{\vartheta}_i + \Psi_{i,1} T_i^{-1} \dot{\vartheta}_i \\
&- Q_{v_i}^*(t) - \sum_{j \in \mathcal{N}_i} P_{ij}^* - D_i w^*.
\end{align*}
\]
\[ \dot{P}_{\text{ref},i}(t) \triangleq P_{\text{ref},i}(t) + T_G, T_{CH}, \Psi_i, T_{i}, \eta_i + (T_G, + T_{CH}, \Psi_i, T_{i}^{-1}) \eta_i \\
+ Q^{\text{ref},i}(t) - \sum_{j \in N_i} P_{ij}^{*} - (D_i + \frac{1}{R_i}) w^* \]

\[ \tilde{P}_{\text{L},i}(t) \triangleq P_{\text{L},i}(t) - \Psi_i, T_{i}^{-1} \eta_i \]

\[ \lambda_i(t) \triangleq \lambda_i(t) + \Psi_i, T_{i}^{-1} \eta_i - c_i, \Psi_i, T_{i}^{-1} \eta_i - b_i \]

\[ \tilde{P}_{ij}(t) \triangleq t_{ij}(\tilde{\theta}_i(t) - \tilde{\theta}_j(t)) - t_{ij}(\theta_i^* - \theta_j^*) \]

\[ \eta_i(t) \triangleq \eta_i(t) - \tilde{\eta}_i(t). \]

(13)

With this coordinate transformation, generator model (1) be

\[ \frac{d\tilde{\theta}_i}{dt} = 2\pi \tilde{w}_i, \]

\[ \frac{d\tilde{w}_i}{dt} = - \frac{1}{m_i}(D_i \tilde{w}_i + \sum_{j \in N_i} \tilde{P}_{ij} - \tilde{P}_{M_i} + \tilde{P}_{E_i}^{*} \]

\[ + \Psi_i, T_{i}^{-1} \eta_i) \]

\[ \frac{d\tilde{P}_{M_i}}{dt} = - \frac{1}{T_{CH}}(\tilde{P}_{M_i} - \tilde{P}_{v_i}), \]

\[ \frac{d\tilde{P}_{v_i}}{dt} = - \frac{1}{T_{G_i}}(\tilde{P}_{v_i} + \frac{1}{R_i} \tilde{w}_i - \tilde{P}_{\text{ref},i}), \]

\[ \frac{d\tilde{P}_{E_i}^{*}}{dt} = c_i \tilde{P}_{E_i} - \lambda_i, \]

(14)

and the internal models (12) be

\[ \dot{\hat{\eta}}_i = \dot{T_i} \Phi_i, T_{i}^{-1} \hat{\eta}_i - N_i(\tilde{P}_{M_i} - \tilde{P}_{E_i}^{*}) \]

(15)

Note that Sylvester equation (11) are used to derive (14) and (15). Let us define system state \[ \hat{x}_i \triangleq [\tilde{\theta}_i \tilde{w}_i \tilde{P}_{M_i} \tilde{P}_{v_i} \tilde{P}_{E_i}^{*}]^T \]

and \[ \hat{z}_i \triangleq [\tilde{\theta}_i^T \tilde{\eta}_i^T]^T \].

Due to the coordinate transformation, frequency regulation problem is transformed to the stabilization problem of the system consisting of (14) and (15). This is because, if (15) is stable, \[ \eta_i \] asymptotically tracks \[ \psi_i \]. Likewise, if (14) and (15) is stable, then system state \[ x_i \] asymptotically follows manifold (9).

D. Coordinate transformation two

To ensure stability of the system (14) and (15), we want to apply input-to-state stability (ISS). However, ISS property can not be directly applied to the system (14) with internal model (15) because the property requires the system is stable. Stability of the system (14) will be insured by coordinate transformation inspired by backstepping [14]. Because the subsystem is recursive structure, we can progressively stabilize outer system \[ \tilde{P}_{v_i} \] to inner system \[ \tilde{\theta}_i \]. For the stability of internal model (15), we have another coordinate transformation on \[ \hat{\eta}_i \]. Consider coordinate transformation described by

\[ \hat{\theta}_i \triangleq \tilde{\theta}_i, \quad \hat{w}_i \triangleq \tilde{w}_i - \tilde{\eta}_i^* \]

\[ \hat{P}_{M_i} \triangleq \tilde{P}_{M_i} - \tilde{P}_{M_i}^{*} \]

\[ \hat{P}_{v_i} \triangleq \tilde{P}_{v_i} - \tilde{P}_{v_i}^{*} \]

\[ \hat{P}_{E_i} \triangleq \tilde{P}_{E_i}^{*} \]

\[ \hat{\lambda}_i \triangleq \hat{\lambda}_i, \]

(16)

where new manifolds are

\[ \hat{u}_i \triangleq \frac{-k_i,1}{2\pi}, \]

\[ \hat{P}_{M_i} \triangleq \left( \sum_{k \in N_i} t_{ij} - \frac{k_i,1}{2\pi}(D_i - m_i, k_i,1 - m_i, \Psi_i, T_{i}^{-1} N_i) \right) \hat{\theta}_i \]

\[ + \left(D_i - m_i, k_i,1 - m_i, k_i,2 - m_i, \Psi_i, T_{i}^{-1} N_i \right) \hat{w}_i \]

\[ \triangleq \frac{e_{M_i,1}}{m_i} \hat{\eta}_i \]

\[ \hat{P}_{v_i} \triangleq e_{M_i,1}(1 - T_{CH} k_i,1) \hat{\theta}_i \]

\[ + (2\pi T_{CH} e_{M_i,1} + e_{M_i,2} - T_{CH} k_i,2 e_{M_i,2}) \hat{w}_i \]

\[ + (1 + \frac{e_{M_i,2}}{m_i} T_{CH} - T_{CH} k_i,3) \hat{P}_{M_i} \]

\[ \triangleq e_{v_i,1} \hat{\theta}_i + e_{v_i,2} \hat{w}_i + e_{v_i,3} \hat{P}_{M_i} \]

\[ \hat{\eta}_i \triangleq -m_i N_i(\hat{w}_i - \frac{k_i,1}{2\pi} \hat{\theta}_i) \]

(17)

where \[ k_{i,n} \] is an arbitrary positive constant. Then the error dynamics (14) with (15) becomes

\[ \frac{d\hat{\theta}_i}{dt} = -k_i,1 \hat{\theta}_i + 2\pi \hat{w}_i, \]

\[ \frac{d\hat{w}_i}{dt} = -k_i,2 \hat{w}_i - \frac{1}{m_i} \left( \sum_{j \in N_i} t_{ij} \hat{\theta}_j - \hat{P}_{M_i}^{*} + \hat{P}_{E_i}^{*} \right) \]

\[ + \Psi_i, T_{i}^{-1} \hat{\eta}_i) \]

\[ \frac{d\hat{P}_{M_i}}{dt} = -k_i,3 \hat{P}_{M_i} + \frac{1}{T_{CH}} \hat{P}_{v_i} \]

\[ - \frac{e_{M_i,2}}{m_i} \left( \sum_{j \in N_i} t_{ij} \hat{\theta}_j - \hat{P}_{E_i}^{*} - \Psi_i, T_{i}^{-1} \hat{\eta}_i \right) \]

\[ \frac{d\hat{P}_{v_i}}{dt} = -k_i,4 \hat{P}_{v_i} - \frac{1}{m_i} (e_{v_i,1} - e_{v_i,2} e_{v_i,3}) \]

\[ \times \left( \sum_{j \in N_i} t_{ij} \hat{\theta}_j - \hat{P}_{E_i}^{*} - \Psi_i, T_{i}^{-1} \hat{\eta}_i \right) + \frac{1}{T_{G_i}} \hat{P}_{E_i}^{*} \]

\[ - \frac{1}{T_{G_i}} \left( e_{v_i,1} \hat{\theta}_i + e_{v_i,2} \hat{w}_i + e_{v_i,3} \hat{P}_{M_i} + e_{v_i,4} \hat{P}_{v_i} \right) \]

\[ \frac{d\hat{P}_{E_i}^{*}}{dt} = -k_i,5 \hat{P}_{E_i}^{*} + (e_i + k_i,5) \tilde{P}_{E_i}^{*} - \lambda_i \]

\[ \hat{\eta}_i = M_i \hat{\eta}_i + \left( \frac{m_i k_{i,1}}{2\pi} M_i + \left( \frac{D_i k_{i,1}}{2\pi} + \sum_{j \in N_i} t_{ij} I \right) N_i \hat{\theta}_j \right) \]

\[ - (m_i M_i + D_i I) N_i \hat{w}_i + \sum_{j \in N_i} t_{ij} \hat{\theta}_j \]

\[ = M_i \hat{\eta}_i + \hat{e}_{v_i,1} \hat{\theta}_i + \hat{e}_{v_i,2} \hat{w}_i + \hat{e}_{v_i,3} \hat{\theta}_j \]

(18)

where

\[ e_{v_i,1} \triangleq e_{v_i,1} - \frac{k_{i,1}}{2\pi R_i} - T_{G_i} k_{i,1} e_{v_i,1} \]

\[ e_{v_i,2} \triangleq e_{v_i,2} + \frac{1}{R_i} + 2\pi T_{G_i} e_{v_i,1} - T_{G_i} k_{i,2} e_{v_i,2} \]

\[ e_{v_i,3} \triangleq e_{v_i,3} + T_{G_i} e_{v_i,2} - T_{G_i} k_{i,3} e_{v_i,3} \]

\[ e_{v_i,4} \triangleq -T_{G_i} k_{i,4} + 1 + \frac{T_{G_i}}{T_{CH}} e_{v_i,3} \]

Let us define the system state as \[ \hat{x}_i \triangleq [\hat{\theta}_i \hat{w}_i \hat{P}_{M_i} \hat{P}_{v_i} \hat{P}_{E_i}^{*}]^T \] and \[ \hat{z}_i \triangleq [\hat{\theta}_i^T \hat{\eta}_i^T]^T \]. We
can rewrite (18) compactly with:
\[
\dot{x}_i = A_i \hat{x}_i + B_i \hat{w}_i + \dot{H}_i \hat{h}_i + \sum_{j \in N_i} \hat{D}_{ij} \hat{x}_j, \\
\dot{\hat{h}}_i = J_i \hat{h}_i + L_i \hat{x}_i + \sum_{j \in N_i} \hat{O}_{ij} \hat{x}_j.
\]
where corresponding matrices can be found in (18). If the system $\hat{z}_i$ in (18) is globally asymptotically stable, $\hat{z}_i$ in (14) and (15) is globally asymptotically stable since the new manifold $\hat{z}_i^*$ in (17) is all zero when $\hat{z}_i$ in (18) stables. Exponential stability of $\hat{z}_i$ in (14) and (15) implies that the original system $x_i$ in (1) asymptotically follows its manifold $x_i^*$ in (9). Therefore, we can guarantee that the original system $x_i$ in (1) asymptotically follows its manifold $x_i^*$ in (9) by guaranteeing the stability of $\hat{z}_i$ (18).

**Algorithm 1** Distributed controller design

for $i \in \mathcal{V}$ do

1) Choose an arbitrary constant $\delta_{i,n_15}$ satisfying (24).
2) Choose a positive constant $k_{i,1}$ satisfying contraction mapping $\gamma_{i,1} < 1$ in (23).
3) Choose a controllable pair $(M_i, N_i)$ with $M_i$ Hurwitz and symmetric.
4) Choose a sufficiently small positive constant $\alpha_i \in \mathbb{R}_{>0}$ satisfying contraction mapping $\alpha_i \gamma_{i,13} < 1, 0 < \alpha_i \gamma_{i,15} < 1$.
5) Replace $N_i$ with $\alpha_i N_i$.
6) Find the solution $T_i$ of Sylvester equation (11) with a new pair $(M_i, \alpha_i N_i)$.
7) Choose constant $k_{i,n_5}$ for $n_5 \in \{2, 3, 4, 5\}$ sequentially which satisfies followings.
\[
\gamma_{i,2}, \ldots, \gamma_{i,13} < 1. 
\]
8) Find following notations sequentially with the set of constant $k_{i,n_5}$ and pair $(M_i, N_i)$:
\[e_{0_{i,n_3}}, e_{M_{i,n_2}}, e_{n_4}, e_{i,n_4}, e_{i,n_4}, \Psi_i, \hat{P}_i, \hat{P}_i, \hat{P}_i, \hat{P}_i.\]
Definition of $\hat{P}, \hat{P}_i$ for $\forall n_5$ is given in (20).
9) With $(M_i, N_i)$, design internal model in (12).
10) With the set of constant $k_{i,n_5}$ and pair $(M_i, N_i)$, design controller (22).

end for

**Definition of $e_{a_{i,n_4}}$ in Algorithm 1** is given by
\[
e_{a_{i,1}} = e_{r_{i,1}} - e_{r_{i,1}} e_{n_4} - e_{M_{i,1}} e_{a_{i,3}} - e_{a_{i,2}} + k_{i,1} e_{a_{i,2}} 2\pi e_{a_{i,2}}
\]
\[
e_{a_{i,2}} = e_{r_{i,2}} - e_{r_{i,2}} e_{a_{i,2}} - e_{M_{i,2}} e_{a_{i,3}} + e_{a_{i,2}} - e_{a_{i,3}}
\]
\[
e_{a_{i,3}} = e_{r_{i,3}} - e_{r_{i,3}} e_{a_{i,4}} + e_{a_{i,4}} - e_{r_{i,4}}.
\]

**Remark 3.1:** We always can select a set of constant $k_{i,n_5}$ for $\forall n_5$ and a pair $(M_i, N_i)$ through Algorithm (1). In 4), $\alpha_i \gamma_{i,13}, \alpha_i \gamma_{i,14}, \alpha_i \gamma_{i,14} \rightarrow 0$ as $\alpha_i \rightarrow 0$ and a pair $(M_i, \alpha_i N_i)$ is controllable if $(M_i, N_i)$ is controllable. In 2) and 7), we can choose sufficiently large $k_{i,n_5}$ for $\forall n_5$ to satisfy contraction mapping respectively.

Now we ready to propose stability theorem.

**Theorem 3.1:** Under Assumption 3.1, 3.2, and 3.3, the networked system having local system dynamics $\hat{z}_i$ described in (18) for $i \in \mathcal{V}$ is globally exponentially stable for any initial state $\hat{z}_i(t_0)$ if the controller is designed through Algorithm 1 with corresponding controller:
\[
\hat{P}_{ref} = e_{r_{i,1}} \hat{\hat{h}}_i + e_{r_{i,3}} \hat{\hat{w}}_i + e_{r_{i,3}} \hat{P}_M + e_{r_{i,4}} \hat{P}_v,
\]
\[
\hat{\lambda}_i = (c_1 + k_{i,5}) \hat{P}_L^E.
\]
Equivalently,
\[
u = \begin{bmatrix} P_{ref} \\ \hat{\lambda}_i \end{bmatrix} = \hat{K}_{i,1} x + \hat{K}_{i,2} \eta + \hat{K}_{i,3} + \hat{K}_{i,4}(t)
\]
where
\[
\hat{K}_{i,1} = \begin{bmatrix} p_{i,11} & p_{i,12} & p_{i,13} & p_{i,14} & p_{i,15} \end{bmatrix},
\]
\[
\hat{K}_{i,2} = \begin{bmatrix} p_{i,21} & p_{i,22} \end{bmatrix}, \hat{K}_{i,3} = \begin{bmatrix} p_{i,31} & p_{i,32} \end{bmatrix}, \hat{K}_{i,4} = \begin{bmatrix} p_{i,41} & p_{i,42} \end{bmatrix}.
\]
Each element is given by
\[
p_{i,11} \triangleq e_{a_{i,1}}, p_{i,12} \triangleq e_{a_{i,2}}
\]
\[
p_{i,13} \triangleq e_{a_{i,3}} + \Psi_i T_i^{-1}(T_G T_C H M_i + T_C H_i (1 - e_{a_{i,4}}) I)
\]
\[
+ T_G T_C H_i N_i \Psi_i T_i^{-1} N_i,
\]
\[
p_{i,14} \triangleq e_{a_{i,4}} + T_G \Psi_i T_i^{-1} N_i,
\]
\[
p_{i,15} \triangleq \Psi_i T_i^{-1}(T_G T_C H_i M_i + (T_C H_i e_{a_{i,4}} - T_G_i)
\]
\[
- T_G_i + T_G T_C H_i k_{i,5}) I - T_G T_C H_i N_i \Psi_i T_i^{-1} N_i,
\]
\[
p_{i,16} \triangleq \Psi_i T_i^{-1} N_i,
\]
\[
p_{i,17} \triangleq e_{a_{i}} + k_{i,5} - \Psi_i T_i^{-1} N_i,
\]
\[
p_{i,21} \triangleq \Psi_i T_i^{-1}(T_G T_C H_i M_i + (T_C H_i e_{a_{i,4}} - T_G_i)
\]
\[
- T_G_i + T_G T_C H_i k_{i,5}) I - T_G T_C H_i N_i \Psi_i T_i^{-1} N_i,
\]
\[
p_{i,22} \triangleq -\Psi_i T_i^{-1} (k_{i,5} I + M_i),
\]
\[
p_{i,31} \triangleq -e_{a_{i}} \theta_i + (D_i q_i + \frac{1}{2} e_{a_{i,1}}) \dot{w}_i + q_i \sum_{j \in N_i} P_{ij}^E
\]
\[
p_{i,32} \triangleq e_{a_{i}} \theta_i^* + \sum_{j \in N_i} P_{ij}^E + \sum_{j \in N_i} P_{ij}^E
\]
\[
p_{i,41} \triangleq e_{a_{i}} \theta_i + (D_i q_i + \frac{1}{2} e_{a_{i,1}}) \dot{w}_i + q_i \sum_{j \in N_i} P_{ij}^E,
\]
\[
p_{i,42} \triangleq -\Psi_i T_i^{-1} N_i P_{ij}^E
\]
where
\[
q_i \triangleq 1 - e_{a_{i,4}} - e_{a_{i,4}} + \Psi_i T_i^{-1} (-T_G T_C H_i M_i)
\]
\[
+ (T_C H_i e_{a_{i,4}} - T_G_i - T_G T_C H_i N_i) \Psi_i T_i^{-1} N_i.
\]

**Proof:** Apply the given controller (21) to the system (18), then $\hat{P}_v$ and $\hat{P}_L^E$ becomes,
\[
\frac{d\hat{P}_v}{dt} = -k_{i,4} \hat{P}_v - e_{v_{i,4}} \sum_{j \in N_i} t_{ij} \hat{\hat{h}}_j - \hat{\hat{P}}_L^E - \Psi_i T_i^{-1} \hat{\hat{h}}_i
\]
\[
\frac{d\hat{P}_L^E}{dt} = -k_{i,5} \hat{\hat{P}}_L^E
\]
where \( e_{v_{1,4}} \triangleq \frac{1}{m} (e_{v_{1,2}} - e_{M_{1,2}} e_{v_{1,3}}) \). Let us have Lyapunov candidate of each agent of \( i \in \mathcal{V}_1 \) as

\[
\begin{align*}
V_{\theta_i} & = \frac{1}{2} \dot{\theta}_i^2, \quad V_{\gamma_i} = \frac{1}{2} \dot{\gamma}_i^2, \quad V_{P_{M_i}} \triangleq \frac{1}{2} \dot{P}_{M_i}^2, \\
V_{\dot{P}_{L_i}} & = \frac{1}{2} \dot{P}_{L_i}^2, \quad V_{P_{E_i}} = \frac{1}{2} \dot{P}_{E_i}^2, \quad V_{\dot{\eta}_i} = \frac{1}{2} \dot{\eta}_i^2
\end{align*}
\]

where each candidate is a positive definite function. Derivative of each Lyapunov candidate is given by

\[
\begin{align*}
\dot{V}_{\theta_i} & = -k_{i,1} \dot{\theta}_i^2 + 2 \pi \dot{\omega}_i \dot{\theta}_i \\
\dot{V}_{\gamma_i} & = -k_{i,2} \dot{\gamma}_i^2 - \frac{1}{m_i} (\sum_{j \in \mathcal{N}_i} t_{ij} \dot{\theta}_j - \dot{P}_{M_i} + \dot{P}_{E_i} - \Psi(t_{ij}^{-1} \dot{\eta}_j) \dot{w}_j) \\
\dot{V}_{P_{M_i}} & = -k_{i,3} \dot{P}_{M_i}^2 + \frac{1}{T_{CH_i}} \dot{P}_{v_i} \dot{P}_{M_i} - \frac{e_{M_{1,2}}}{m_i} (\sum_{j \in \mathcal{N}_i} t_{ij} \dot{\theta}_j - \dot{P}_{L_i} - \Psi(t_{ij}^{-1} \dot{\eta}_j)) \dot{P}_{v_i} \\
\dot{V}_{\dot{P}_{E_i}} & = -k_{i,4} \dot{P}_{E_i}^2 - \frac{1}{m} (e_{v_{1,2}} - e_{M_{1,2}} e_{v_{1,3}}) \\
& \quad \times (\sum_{j \in \mathcal{N}_i} t_{ij} \dot{\theta}_j - \dot{P}_{L_i} - \Psi(t_{ij}^{-1} \dot{\eta}_j)) \dot{P}_{v_i} \\
\dot{V}_{\dot{\eta}_i} & = \frac{1}{2} \dot{\eta}_i T (M_i \dot{\eta}_i + \dot{e}_{n_{i,1}} \dot{\theta}_i + \dot{e}_{n_{i,2}} \dot{w}_i + \dot{e}_{n_{i,3}} \dot{\theta}_j) \\
& \quad + \frac{1}{2} (M_i \dot{\eta}_i + \dot{e}_{n_{i,1}} \dot{\theta}_i + \dot{e}_{n_{i,2}} \dot{w}_i + \dot{e}_{n_{i,3}} \dot{\theta}_j) T \dot{\eta}_i.
\end{align*}
\]

This is equivalent to ISS property [12]:

\[
\begin{align*}
|\dot{\theta}_i| & \leq \max \left\{ e^{-k_{i,1} (1-\delta_{i,1}) (t-t_0)} |\dot{\theta}_i(t_0)|, \gamma_{i,1} |\dot{w}_i(t_0)| \right\} \\
|\dot{w}_i| & \leq \max \left\{ e^{-k_{i,2} (1-\delta_{i,2} - \delta_{i,3} - \delta_{i,4} - \delta_{i,5}) (t-t_0)} |\dot{w}_i(t_0)|, \gamma_{i,2} \max_{j \in \mathcal{N}_i} |\dot{\theta}_j(t_0)|, \gamma_{i,3} |\dot{P}_{M_i}(t_0)|, \gamma_{i,4} |\dot{P}_{E_i}(t_0)|, \gamma_{i,5} |\dot{\eta}_i(t_0)| \right\} \\
|\dot{\eta}_i| & \leq \max \left\{ e^{-k_{i,6} (1-\delta_{i,6} - \delta_{i,7} - \delta_{i,8} - \delta_{i,9}) (t-t_0)} |\dot{\eta}_i(t_0)|, \gamma_{i,6} |\dot{P}_{v_i}(t_0)|, \gamma_{i,7} |\dot{V}_{\theta_i}(t_0)|, \gamma_{i,8} |\dot{V}_{\gamma_i}(t_0)|, \gamma_{i,9} |\dot{V}_{P_{M_i}}(t_0)|, \gamma_{i,10} |\dot{V}_{\dot{P}_{E_i}}(t_0)| \right\} \\
|\dot{\eta}_i| & \leq \max \left\{ e^{-k_{i,11} (1-\delta_{i,11} - \delta_{i,12}) (t-t_0)} |\dot{\eta}_i(t_0)|, \gamma_{i,11} |\dot{\theta}_i(t_0)|, \gamma_{i,12} |\dot{P}_{v_i}(t_0)|, \gamma_{i,13} |\dot{V}_{\theta_i}(t_0)|, \gamma_{i,14} |\dot{V}_{\gamma_i}(t_0)|, \gamma_{i,15} \max_{j \in \mathcal{N}_i} |\dot{\theta}_j(t_0)| \right\}
\end{align*}
\]

where

\[
\begin{align*}
\gamma_{i,1} & \triangleq \frac{2 \pi}{k_{i,1} \delta_{i,1}}, \quad \gamma_{i,2} \triangleq \sum_{j \in \mathcal{N}_i} t_{ij} m_i / k_{i,2} \delta_{i,2}, \quad \gamma_{i,3} \triangleq 1 / m_i, \quad \gamma_{i,4} \triangleq \frac{1}{k_{i,2} \delta_{i,3}}, \\
\gamma_{i,5} & \triangleq \frac{\| \Psi(t_{ij}^{-1}) \| / m_i}{k_{i,4} \delta_{i,4}}, \quad \gamma_{i,6} \triangleq \frac{1}{T_{CH_i}} / k_{i,5} \delta_{i,5}, \\
\gamma_{i,7} & \triangleq \frac{e_{M_{1,2}} \sum_{j \in \mathcal{N}_i} t_{ij} m_i}{k_{i,7} \delta_{i,7}}, \quad \gamma_{i,8} \triangleq \frac{e_{M_{1,2}} \sum_{j \in \mathcal{N}_i} t_{ij} m_i}{k_{i,8} \delta_{i,8}}
\end{align*}
\]

and \( \delta_{i,q} \) for \( q \in \{1, \ldots, 15\} \) is a positive constant satisfying

\[
\begin{align*}
0 & < 1 - \delta_{i,1} < 1, \\
0 & < 1 - \delta_{i,2} - \delta_{i,3} - \delta_{i,4} - \delta_{i,5} < 1, \\
0 & < 1 - \delta_{i,6} - \delta_{i,7} - \delta_{i,8} - \delta_{i,9} < 1, \\
0 & < 1 - \delta_{i,10} - \delta_{i,11} - \delta_{i,12} < 1, \\
0 & < 1 - \delta_{i,13} - \delta_{i,14} - \delta_{i,15} < 1.
\end{align*}
\]

Proof: It is a direct result of Theorem 3.1 and coordinate transformation (13) and (16).

Note that requirements, contraction mapping, for control parameters in Theorem 3.1 and 3.2 are sufficient condition for stability of the system.

IV. SIMULATION

Consider 4-bus grid network having a ring topology 1-2-3-4-1. A bus is connected to two other buses with power-line.

A. Parameters and Matrices

Following parameters adopted from page 598 in [15]

\[
\begin{align*}
m_1 & = 10 s, \quad D_1 = 1 MW/Hz, \quad R_1 = 0.05 H/\Omega, \\
T_{CH_i} & = 0.3 s, \quad T_{G_i} = 0.2 s, \quad t_{ij} = 1.5 MW/\text{rad}
\end{align*}
\]

We set the manifolds as \( w_{i}^* = 60 H/\Omega \) for \( \forall i \) with

\[
\begin{align*}
\theta_i^* = 5 \text{ rad} \quad \theta_i^2 = 4 \text{ rad} \quad \theta_i^3 = 3 \text{ rad} \quad \theta_i^4 = 2 \text{ rad}.
\end{align*}
\]

Partial information about wind power generation is given by

\[
\begin{align*}
\Psi_{i,1} & = [30 \ 0 \ 0 \ 0], \quad \Psi_{i,2} = [0 \ 0 \ 30 \ 0], \\
\rho_i^L & = 0.002 \text{ rad}, \quad \rho_i^M = 0.02 \text{ rad}.
\end{align*}
\]

We have chosen control parameters as

\[
\begin{align*}
k_{i,1} & = 6, \quad k_{i,2} = 6, \quad k_{i,3} = 40, \quad k_{i,4} = 50, \quad k_{i,5} = 5, \\
M_i & = \begin{bmatrix}
-5 & -1 & -1 & 1 \\
-1 & -5 & -1 & 4 \\
-1 & -1 & -5 & 1 \\
1 & 4 & 1 & -5
\end{bmatrix}, \quad N_i = \begin{bmatrix}
1 \\
3 \\
-4 \\
1
\end{bmatrix}.\n\end{align*}
\]
In elastic power demand is $P^E = 160 + 30 \sin(0.008t)$ with initial condition $\hat{x}_i = [-1 \ 1 \ 10 \ 10 \ 10]^T$, $\hat{y}_i = [1 \ 1 \ 1 \ 1 \ 1]^T$.

**B. Result**

![Figure 1: Error $\|\hat{z}(t)\|$](image)

Figure 1 shows evolution of error $\|\hat{z}(t)\|$ where $\hat{z} \triangleq [\hat{z}_1^T(t), \hat{z}_2^T(t), \hat{z}_3^T(t), \hat{z}_4^T(t)]T(t)$. Error $\|\hat{z}(t)\|$ converges to zero asymptotically. After short transient part, mechanical power generation $P_M(t)$ and elastic power demand $P^E_M(t)$ asymptotically track actual $P^*_M(t)$ and $P^*_E(t)$ given in (25), a combination of sinusoidal functions, in (9) respectively.

$$
P^*_M = 224.5 + 30 \sin(0.008t) - 30 \sin(0.02t)
$$

$$
P^*_E = 30 \sin(0.008t)
$$

(25)

**V. CONCLUSIONS**

We have proposed a distributed controller which achieves global frequency stability by adopting internal model principles to cope with uncertainty caused by renewable energy generation. We separate low and medium frequency part of renewable energy and assigned them to each controller: reference input of synchronous generator and pricing control. The approach used in this paper can be applied to the system having other types of renewable energy or multi-generator system splitting frequencies finer.

**VI. APPENDIX**

Consider an interconnected system described by

$$
\dot{x}_i(t) = f_i(x(t), u_i(t), d(t))
$$

where $x(t)$, $u(t)$, and $d(t)$ denote system state, input and uncertainty respectively.

**Assumption 6.1:** The system (26) is input-to-state stable with proper input $u_i$. Equivalently, there exists local controller $u_i$, class $\mathcal{KL}$ function $\beta$ and class $\mathcal{K}$ function $\gamma_{id}$ and $\gamma_{ij}$ such that

$$
\|x_i(t)\| \leq \max\{\beta(\|x_i(t_0)\|, t - t_0), \gamma_{id}(\|d(t)\|, t_0, t)\},
\max_{j \in \mathcal{N}_i}\{\gamma_{ij}(\|x_j\|, t_0, t)\}\}
$$

(27)

for all $t \geq t_0$.

**Assumption 6.2:** Gain functions $\gamma_{ij}$ are contraction mappings for $(i, j) \in \mathcal{E}$; i.e., $\gamma_{ij}(s) < s$ for all $s > 0$.

**Theorem 6.1:** (Distributed constrained small gain theorem) Under assumption 6.1 and 6.2, the system (26) is ISS stable with respect to $d$. Equivalently, there exists class $\mathcal{KL}$ function $\beta$ and class $\mathcal{K}$ function $\gamma_{id}$ such that

$$
\|x(t)\| \leq \max\{\beta(\|x(t_0)\|, t - t_0), \gamma_{id}(\|d(t)\|, t_0, t)\},
$$

(28)

for any initial state $x_i(t_0)$ for all $t \geq t_0$.

**Proof:** For the notational simplicity in the sequel proof, we assume that $V$ is complete; i.e., $\mathcal{N}_i = \mathcal{V} \setminus \{i\}$. If $(i, j) \not\in \mathcal{E}$, then $\gamma_{ij}(s) = s$ and $\Delta_{ij} = +\infty$. We divide the remaining of the proof into three claims.

**Claim 1:** We will prove by induction that the following hold for $i \in \{1, \ldots, \ell\}$:

$$
\|x_i\|_{[t_0, T]} \leq \max\{\beta_1(\|x_1(t_0)\|, 0), \gamma_{id}(\|d_1\|, [t_0, T])\},
\max_{(i_1, i_2, \ldots, i_n) \in \mathcal{P}_{ij}}\gamma_{i_1 i_2 \cdots i_n} \circ \cdots \circ \gamma_{i_{n-1} i_n} \circ \gamma_{i_n d}(\|d_{i_n}\|, [t_0, T])
$$

(29)

**Proof:** By (27), one can see that

$$
\|x_1\|_{[t_0, T]} \leq \max\{\beta_1(\|x_1(t_0)\|, 0), \gamma_{id}(\|d_1\|, [t_0, T])\},
\max_{j \neq 1}\{\gamma_{1j}(\|x_j\|, [t_0, T])\}
$$

(30)

and

$$
\|x_2\|_{[t_0, T]} \leq \max\{\beta_2(\|x_2(t_0)\|, 0), \gamma_{id}(\|d_2\|, [t_0, T])\},
\max_{j \neq 2}\{\gamma_{2j}(\|x_j\|, [t_0, T])\}
$$

(31)

Substitute (31) into (30), and it renders the following:

$$
\|x_1\|_{[t_0, T]} \leq \max\{\beta_1(\|x_1(t_0)\|, 0), \gamma_{id}(\|d_1\|, [t_0, T])\},
\gamma_{12} \circ \beta_2(\|x_2(t_0)\|, 0), \gamma_{1d}(\|d_2\|, [t_0, T]),
\max_{j \neq 2}\{\gamma_{1j}(\|x_j\|, [t_0, T])\}
$$

(32)

Since $\gamma_{12} \circ \gamma_{21}$ is a contraction mapping, it follows from (32) that

$$
\|x_1\|_{[t_0, T]} \leq \max\{\beta_1(\|x_1(t_0)\|, 0), \gamma_{id}(\|d_1\|, [t_0, T])\},
\gamma_{12} \circ \beta_2(\|x_2(t_0)\|, 0), \gamma_{1d}(\|d_2\|, [t_0, T]),
\max_{j \neq 1, 2}\{\gamma_{1j}(\|x_j\|, [t_0, T])\}
$$

(33)
By symmetry, one can show a similar property to (33) for $\|x_2\|_{t_0,T}$. So (29) holds for the case of $\ell = 2$.

Now assume that (29) holds for some $\ell < n$. Similar to (30), we have

$$
\|x_{\ell+1}\|_{t_0,T} \leq \max\{\beta_{\ell+1}(\|x_{\ell+1}(t_0)\|, 0), 
\gamma(\ell+1)d(\|x_{\ell+1}\|_{t_0,T}), 
\max_{j \neq (\ell+1)} \{\gamma_{(\ell+1)j}(\|x_j\|_{t_0,T})\}\}. \quad (34)
$$

Following analogous steps above, one can show that (30) holds for $\ell + 1$. By induction, we complete the proof.

**Claim 2:** We show by contradiction the existence and boundedness of the solution to (26).

**Proof:** A direct result of Claim 1 is that the following holds for all $i \in V$:

$$
\|x_i\|_{t_0,T} \leq \max\{\beta_0(\|x_i(t_0)\|, 0), 
\max_{(i_1,i_2,\cdots,i_k) \in P_i} \{\gamma_{i_1i_2}\cdots\gamma_{i_k}(\|d_{i_k}\|_{t_0,T})\}, 
\max_{j \neq i} \max_{(j_1,j_2,\cdots,j_k) \in P_j} \{\gamma_{j_1j_2}\cdots\gamma_{j_k}(\|x_{j_k}(t_0)\|, 0)\}\}. \quad (35)
$$

Since all the gain functions $\gamma_{ij}$ are contraction mappings, (35) renders the following:

$$
\|x_i\|_{t_0,T} \leq \max\{\beta_0(\|x_i(t_0)\|, 0), 
\max_{j \neq i} \gamma_{ij}\circ\gamma_{jd}(\|d_j\|_{t_0,T}), 
\max_{j \neq i} \beta_j(\|x_j(t_0)\|, 0)\}. \quad (36)
$$

Because of the choice of $x_i(t_0)$ and the bound on $d$, the relation (35) holds for any $T$. It implies that $x_i(t) \in X_i$ for all $t \geq t_0$ and thus is uniformly bounded. It completes the proof.

**Claim 3:** We show the ISS property of (26).

We will prove by induction that the following holds for all $i \in S_1 = \{1, \cdots, \ell\}$:

$$
\|x_i(t)\| \leq \max\{\beta_i^{(\ell-1)}(\|x_i\|_{t_0,t}, t - t_0), 
\max_{j \neq i} \max_{(j_1,j_2,\cdots,j_k) \in P_j} \{\gamma_{j_1j_2}\cdots\gamma_{j_k}(\|x_{j_k}(t_0)\|, 0)\}\};
$$

for some class $\mathcal{KL}$ function $\beta_i^{(\ell-1)}$.

Let $\ell = 2$. Note that

$$
\|x_1(t_0 + T)\| \leq \max\{\beta_1(\|x_1(t_0 + T)\|, t - t_0), 
\gamma_{1d}(\|d_1\|_{t_0,T}, t - t_0), 
\max_{j \neq 1} \gamma_{1j}(\|x_j\|_{t_0,T}, t - t_0)\};
$$

for some class $\mathcal{KL}$ function $\beta_1^{(\ell-1)}$.

By symmetry, there is class $\mathcal{KL}$ function $\tilde{\beta}_2$ such that

$$
\|x_2(t_0 + T)\| \leq \max\{\tilde{\beta}_2(\|x_2(t_0 + T)\|, t - t_0), 
\gamma_{2d}(\|d_2\|_{t_0,T}, t - t_0), 
\max_{j \neq 2} \gamma_{2j}(\|x_j\|_{t_0,T}, t - t_0)\};
$$

Hence, we have shown that (37) holds for $\ell = 2$.

Now assume (37) holds for some $\ell < n$. Recall that

$$
\|x_{\ell+1}(t)\| \leq \max\{\beta_{\ell+1}(\|x_{\ell+1}(t)\|, t - t_0), 
\gamma_{\ell+1}(\|d_{\ell+1}\|_{t_0,t}, t - t_0), 
\max_{j \neq \ell+1} \gamma_{\ell+1j}(\|x_j\|_{t_0,t}, t - t_0)\};
$$

By using similar arguments towards the case of $n = 2$, one can show (37) holds for $\ell + 1$.

Now we proceed to find a relation between $\|x\|_\infty$ and $\|d\|_\infty$. Note that

$$
\|x_i\|_\infty \leq \max\{\beta_i(\|x_i\|, 0), \gamma_{id}(\|d_i\|, 0), 
\max_{j \neq i} \gamma_{ij}(\|x_j\|_\infty)\}.
$$
Similar to (37), one can show by induction that there are class $\mathcal{KL}$ functions $\rho_i$ and $\rho_d$ such that
\[
\|x_i\|_\infty \leq \max\{\rho_i(\|x_i(t_0)\|), \rho_d(\|d_i\|_{t_0,t})\}.
\] (45)

The combination of (45) and (37) achieves the desired result.

REFERENCES


